Nonlinear Systems

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Chapter 1

Introduction to theory: basic definitions

A general nonlinear dynamic system may be modeled by a finite number of coupled first-order ordinary differential equations.

$$\dot{x} = f(t, x, u) \tag{1}$$

where $x = [x_1, \cdots, x_n]^T \in \mathbb{R}^n$: state vector, **n**: system order,
$u = [u_1, \cdots, u_m]^T \in \mathbb{R}^m$: input vector (control, disturbance).

- **f(.)** is a vector field in $\mathbb{R}^n$: a function associating a vector to n-dim point **x**.
- **Initial condition**: $x(t_0) = (x_1(0), x_2(t_0), \cdots, x_n(t_0))$
- Eq. (1) is called **state equation**.
- **Solution** is in the form $x(t_0, t)$ that defines a family of time trajectories in the state space (also referred as phase space).Imposing the initial condition $x(t_0)$ determines one unique trajectory.
- Another equation named **output equation**:

$$y = h(t, x, u) \tag{2}$$

- where $y = [y_1, \cdots, y_p]^T \in \mathbb{R}^p$: output vector.
- Eqs. (1) and (2) together are called **state-space model**.

Introduction to theory: basic definitions

Analysis : unforced state equations

$$\dot{x} = f(x), \quad f : \mathbb{R}^n \to \mathbb{R}^n \quad \text{time-invariant (autonomous)}$$
$$\dot{x} = f(t, x), \quad f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \quad \text{time-varying (Non-autonomous)}$$

Control design:

$$\dot{x} = f(x, u), \quad f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad \text{autonomous with inputs}$$
$$\dot{x} = f(t, x, u), \quad f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \quad \text{Non-autonomous with inputs}$$
$$\dot{x} = f(x) + g(x)u, \quad f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^m \to \mathbb{R}^n \quad \text{affine in } u$$

Linear systems: if f and h are linear functions of x and u

$$\dot{x} = f(x) \quad \to \quad \dot{x} = Ax \quad \text{autonomous}$$
$$\dot{x} = f(x, u) \quad \to \quad \dot{x} = Ax + Bu$$

$$\dot{x} = f(t, x, u) \quad \to \quad \dot{x} = A(t)x + B(t)u$$

Introduction to theory: basic definitions

For linear systems

The Linear Systems satisfy the superposition principals:

1. Homogeneity: $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in \mathbb{R}$.
2. Additivity: $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$.

Unique equilibrium point.

Stability is independent of initial conditions $x(t) = x(0)e^{At}$

Introduction to theory: basic definitions

For nonlinear systems

Non superposition principle $\Longrightarrow$ More complex behavior

Example : Under-water vehicle $\dot{v} + |v|v = u$

Settles faster in response to positive step.

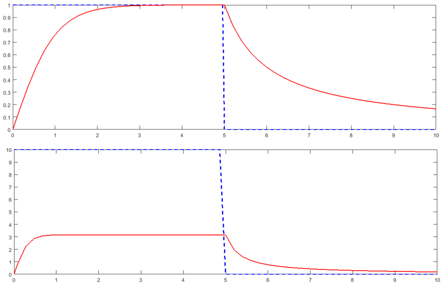
Scaling input does not result into the same scaling in output.

$u = 1$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 1 \\ \Rightarrow v_s &= 1 \end{aligned}$$

$u = 10$

$$\begin{aligned} \Rightarrow 0 + |v_s|v_s &= 10 \\ \Rightarrow v_s &= \sqrt{10} \end{aligned}$$



Nonlinear System Examples

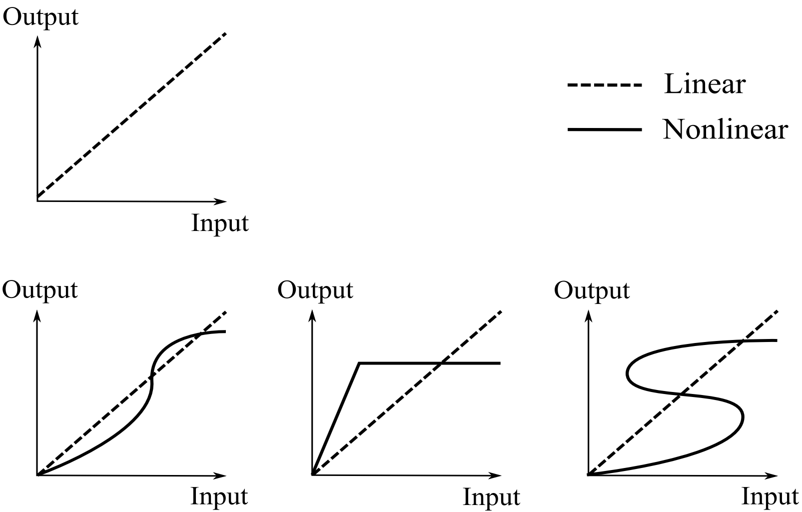


Figure: A nonlinear system is a system in which the change of the output is not proportional to the change of the input.

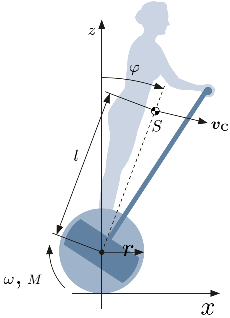
Nonlinear System Examples

Systems with essential nonlinearities in the model

Self-balancing vehicle

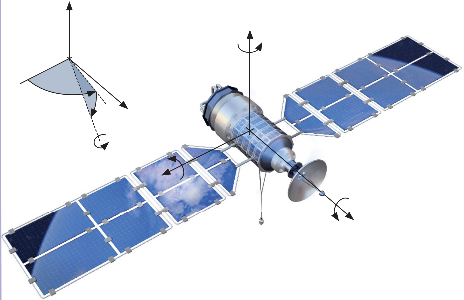
$$\ddot{x} = \frac{c}{a}\ \ \dot{\varphi}^2\sin(\varphi) - \ddot{\varphi}\cos(\varphi)\big) + \frac{M}{ar},$$

$$\ddot{\varphi} = \frac{c}{b}\left(g\sin(\varphi) - \ddot{x}\cos(\varphi)\right) - \frac{M}{b},$$



Euler's rotation equations

$$J_x\dot{\omega}_x = -(J_z - J_y)\omega_y\omega_z + M_x$$
$$J_y\dot{\omega}_y = -(J_x - J_z)\omega_x\omega_z + M_y$$
$$J_z\dot{\omega}_z = -(J_y - J_x)\omega_x\omega_y + M_z$$

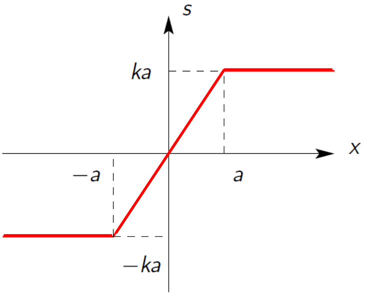


Nonlinear System Examples

Systems with saturation

$$\begin{aligned} \dot{x} &= Ax + Bsat(u) \\ u &= PID(x) \end{aligned}$$

$$sat(u) = \begin{cases} u & \text{If} \quad |u| \leq 1 \\ sgn(u) & \text{If} \quad |u| \geq 1 \end{cases}$$

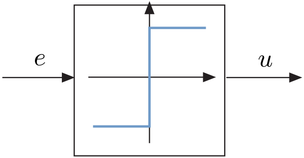


The output is proportional to input for a limited range.

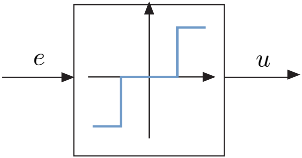
Output becomes constant if input is outside this range.

Common Nonlinearities

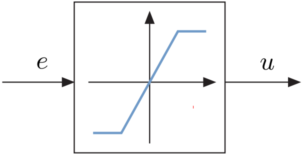
memoryless



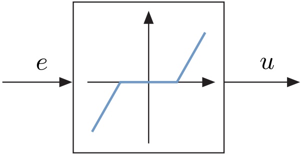
Two-position element



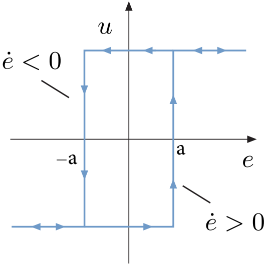
Three-position element



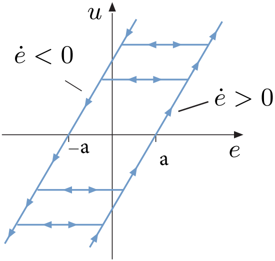
(Saturation characteristic)



Dead zone



Hysteresis characteristic curve



Backlash characteristic curve

Equilibrium points:

An **equilibrium point** represents a stationary condition of a dynamical system. The state $x_e \in \mathbb{R}^n$ is a fixed point for $\dot{x} = f(x)$ if $f(x_e) = 0$, $\forall t \geq 0$.

If a dynamical system has an initial condition $x(0)$ at an equilibrium point x_e, then it will stay at x_e forever, i.e. $x(t) = x_e$, $\forall t \geq 0$.

Example 1: $\dot{x} = x(x-2)^2$

$$f(\bar{x}) = 0 \Rightarrow x(x-2)^2 = 0 \Rightarrow \begin{cases} x_e &= 0 \\ x_e &= 2 \end{cases}$$

This system has two isolated equilibrium points at 0 and 2.

Example 2:

$$\dot{x} = \sin(x), \quad x(0) = x_0 \in \mathbb{R} \tag{3}$$

$x(t) \in \mathbb{R}$ 1st order system (scalar state)

$$f(\bar{x}) = \sin(\bar{x}) = 0 \Longrightarrow \bar{x} = k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$$

This system has infinite many equilibrium points.

Equilibrium points:

Non-isolated equilibria

For a linear system $\dot{x} = Ax, \qquad A \in \mathbb{R}^{n\times n}$.

If A is nonsingular (det(A)$\neq 0$), then $x^* = 0$ is the unique equilibrium.

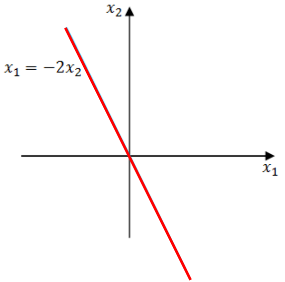
If A is singular (det(A)$= 0$), then Null space defines a continuum of equilibria.

Example:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

$$\dot{x} = 0 \Rightarrow Ax = 0 \Rightarrow \begin{cases} -2x_1 - 4x_2 & = & 0 \\ -x_1 - 2x_2 & = & 0 \end{cases}$$

$$\Rightarrow x_1 = -2x_2$$



Linear approximations around an equilibrium

A linear approximation of a system around an equilibrium point can be used to study the behavior of a nonlinear system around the equilibrium point.

Local stability properties of x_e can be determined by linearizing the vector field $f(x)$ at x_e:

$$f(x_e + \tilde{x}) = \underbrace{f(x_e)}_{=0} + \underbrace{\left.\frac{\delta f}{\delta x}\right|_{x=\bar{x}}}_{\triangleq A} \tilde{x} + \text{higher order terms in } (\mathrm{x} - \mathrm{x_e}) \quad (4)$$

Thus, the linearized model is :

$$\dot{\tilde{x}} = A\tilde{x} \quad (5)$$

The solution of the linearized form (5) has the form : $\tilde{x} = e^{At}x_0$

- If $\mathrm{Re}(\lambda_i(A)) < 0$, then x_e is locally asymptotically stable.
- If $\mathrm{Re}(\lambda_i(A)) > 0$ for some eigenvalue λ_i of A, then $\bar{x}$ is unstable.

Linear approximations around an equilibrium

Example 1: Linearization of $\dot{x} = \sin(x)$

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \cos(x)|_{\bar{x}} = \begin{cases} \cos(2K\pi) & = & 1, & n \text{ even} \\ \sin((2K+1)\pi) & = & -1, & n \text{ odd} \end{cases}$$

$\bar{x} = n\pi,\ n = 0, \pm 1, \pm 2, \cdots$.

Example 2: (2nd order system)

$$\dot{x} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + x_1^2 x_2 - x_2 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x_1 x_2 & x_1^2 - 1 \end{bmatrix}$$

Linearization around $x_e(0,0)$: $\left.\frac{\delta f}{\delta x}\right|_{x_e} (x - x_e) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Caveats:

1. Only local properties can be determined from the linearization.
2. If $Re(\lambda_i) \leq 0$ with some e-values having $Re(\lambda_i) = 0$, then linearization is inconclusive as a stability test. Higher order terms determine stability.

Example: $(a) : \dot{x} = x^3, \qquad (b) : \dot{x} = -x^3.$
In both cases, linearized systems around $\bar{x} = 0$ are the same: $\dot{x} = 0 \Rightarrow x(t) = x_0$, but NL systems have different behaviors.