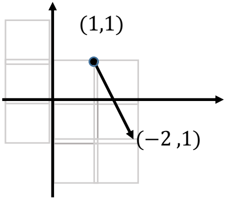
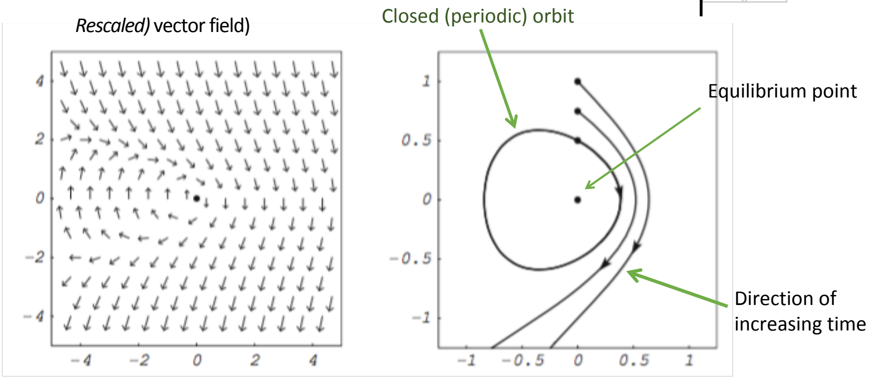
Vector fields and Orbits

Example2:

$$\left\{\begin{array}{lllll} \dot{x} & = & y & = & f_x(x,y) \\ \dot{y} & = & -x-y^2 & = & f_y(x,y) \end{array}\right. \qquad \begin{array}{lll} (x,y) & \rightarrow & f(x,-x-y^2) \\ (1,1) & \rightarrow & (1,-2) \end{array}$$

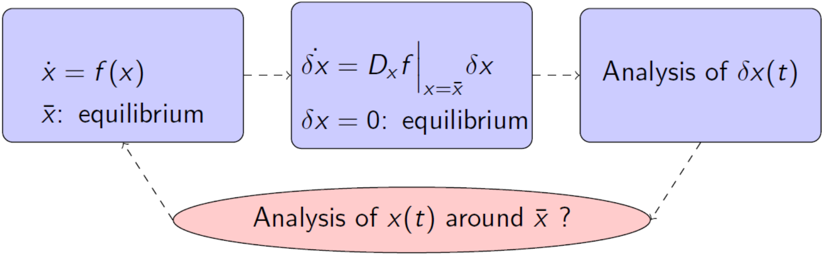




Phase plan analysis

Problem

When $x(t) \in \mathbb{R}^2$, study state trajectories around an equilibrium state



Behavior of Linear second Order Systems

Consider the following linear system

$$\dot{x} = Ax, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

$(a, b, c, d) \in \mathbb{R}$. Change of coordinates: $z(t) = T^{-1}x(t)$, $T \in \mathbb{R}^{2\times 2}$ invertible.

$$\dot{z}(t) = T^{-1}\dot{x}(t) = T^{-1}Ax(t) = T^{-1}ATz(t) = Jz(t)$$

The system $\dot{z} = Jz$ is equivalent to the system $\dot{x} = Ax$.

Remark

A and $J = T^{-1}AT$ are similar $\Longrightarrow$ they have the same eigenvalues

One can always choose T such that J is in real Jordan form

the new coordinates are called normal

Behavior of Linear second Order Systems

There are three possible Jordan forms for A:

- Different real eigenvalues.
- Equal real eigenvalues.
- Complex conjugate.

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \qquad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} \qquad \begin{bmatrix} \alpha & \beta \\ -\beta & \lambda \end{bmatrix}$$

where k=0 or 1.

- In addition, we need to consider the case where at least one of the eigenvalues is zero.

Behavior of Linear second Order Systems

Case 1: $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\lambda_1 \in \mathbb{R}$, and independent eigenvectors

In this case

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \Longrightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively.

The change of coordinate $z = T^{-1}x$, transforms the system into two decoupled first-order differential equations, i.e.,

$$\dot{z}_1 = \lambda_1 z_1, \qquad\qquad \dot{z}_2 = \lambda_1 z_2$$

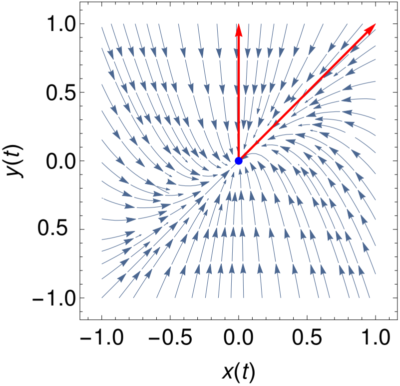
with solution

$$z_1(t) = z_{10} e^{\lambda_1 t}, \quad z_2(t) = z_{20} e^{\lambda_2 t} \Longrightarrow z_2(t) = \frac{z_{20}}{(z_{10})^{\lambda_2/\lambda_1}} z_1^{\lambda_2/\lambda_1}$$

Behavior of Linear second Order Systems

Case 1a: $\lambda_1 < 0$ and $\lambda_2 < 0$

The origin is called **stable node**



Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -6x - 2y \\ \dot{y} &= -2x - 9y \end{aligned}$$

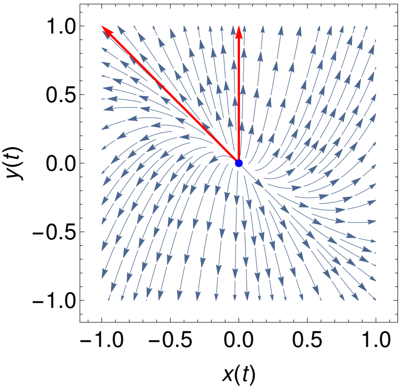
eigenvalues : $\lambda_1 = -10; \lambda_2 = -5 \Longrightarrow$ *stable node*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix},$

Behavior of Linear second Order Systems

Case 1b: $\lambda_1 > 0$ and $\lambda_2 > 0$

The origin is called **Unstable Node**



Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= x + 4y \end{aligned}$$

eigenvalues : $\lambda_1 = 2; \lambda_2 = 3 \Longrightarrow$ *unstable node*

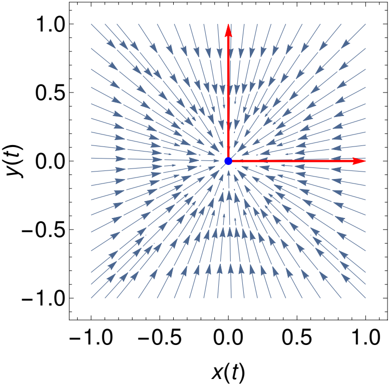
eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

Behavior of Linear second Order Systems

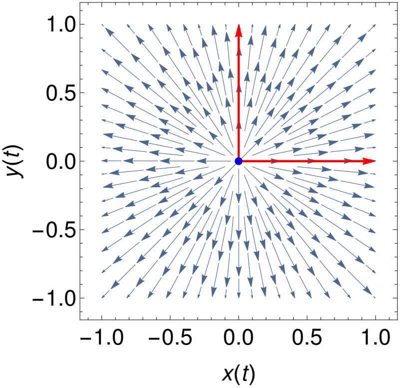
Case 1d: $\lambda_1 = \lambda_2$

The origin is called **stable/unstable degenerate node**

Stable degenerate node: $\lambda_1 = \lambda_2 < 0$



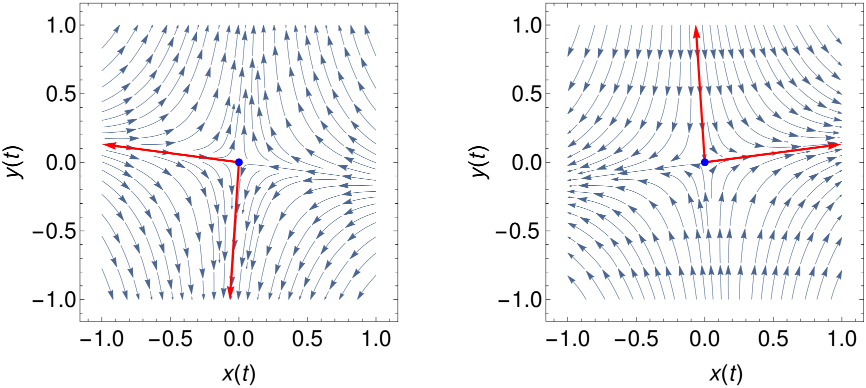
Unstable degenerate node: $\lambda_1 = \lambda_2 > 0$



Behavior of Linear second Order Systems

Case 1b: $\lambda_1 < 0 < \lambda_2$

The origin is called **saddle**



Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= 3x + 4y \\ \dot{y} &= x \end{aligned}$$

eigenvalues : $\lambda_1 = 4; \lambda_2 = -1 \Longrightarrow$ *Saddle*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$

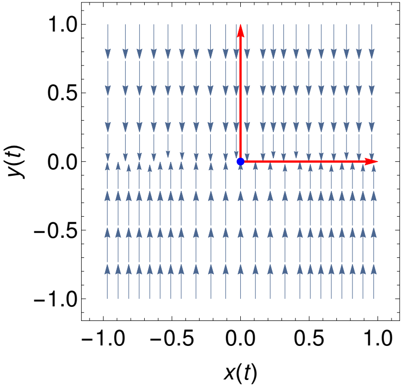
$$\dot{z}_1 = \lambda_1 z_1 \rightarrow \dot{z}_1(t) = z_1(0)e^{\lambda_1 t}$$
$$\dot{z}_2 = \lambda_2 z_2 \rightarrow \dot{z}_2(t) = z_2(0)e^{\lambda_2 t}$$

Case 1e: degenerate saddle

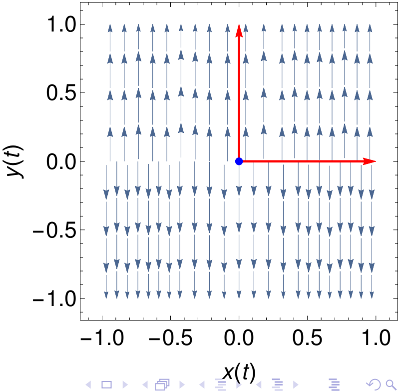
$\lambda_1 < \lambda_2 = 0 \rightarrow$ all states on the z_2 axis are equilibrium states.

$0 = \lambda_1 < \lambda_2 \rightarrow$ all states on the z_1 axis are equilibrium states.

$\lambda_1 < \lambda_2 = 0$



$0 = \lambda_1 < \lambda_2$



Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= 3x - y \\ \dot{y} &= -3x + y \end{aligned}$$

eigenvalues : $\lambda_1 = 4; \lambda_2 = 0 \Longrightarrow$ *degenerate Source*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 0.3333 \\ 1 \end{pmatrix}$

Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= x - 2y \\ \dot{y} &= -2x - 4y \end{aligned}$$

eigenvalues : $\lambda_1 = -5; \lambda_2 = 0 \Longrightarrow$ *Degenerate Sink*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Behavior of Linear second Order Systems

Case 2: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ $\lambda \in \mathbb{R}$, One can show that the state trajectories are given by

$$\begin{aligned} \dot{z}_1 &= z_{10}e^{\lambda t} + z_{20}te^{\lambda t} \\ \dot{z}_2 &= z_{20}e^{\lambda t} \end{aligned} \tag{3}$$

Assume $z_{20} \neq 0$. If $\lambda \neq 0$, from (3-b-) one gets

$$e^{\lambda t} = \frac{z_2(t)}{z_{20}} \Longrightarrow t = \frac{1}{\lambda} \ln \left(\frac{z_2(t)}{z_{20}} \right)$$

and using (3-a-) one obtains

$$z_1(t) = z_{10}\frac{z_2(t)}{z_{20}} + \frac{1}{\lambda} \ln \left(\frac{z_2(t)}{z_{20}} \right) z_2(t)$$

Behavior of Linear second Order Systems

Case 2: $\lambda \neq 0$

The origin is called **stable/unstable improper node**

Only the z_1 axis is invariant.

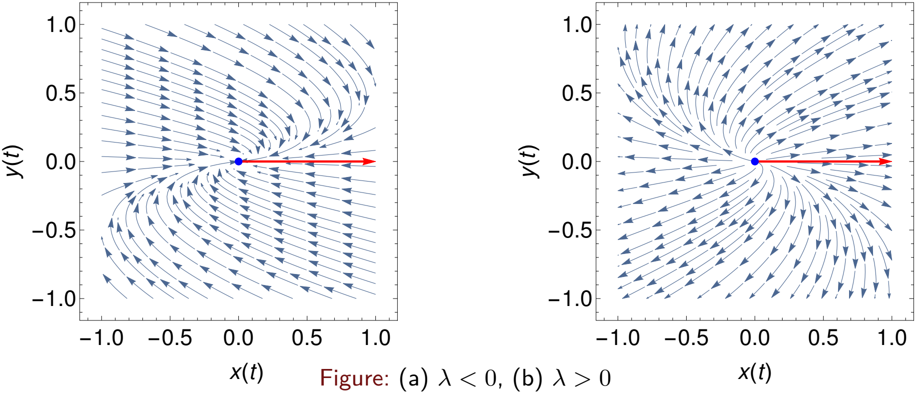


Figure: (a) $\lambda < 0$, (b) $\lambda > 0$

Behavior of Linear second Order Systems

Case 3: Complex conjugate eigenvalues $\lambda_{1,2} = \alpha \pm j\beta \in \mathbb{C}$

Let $\mathbf{v}_1 = u + jv$, $\mathbf{v}_2 = u - jv$ be the eigenvectors associated to the eigenvalues $\lambda_1 = \alpha + j\beta$, $\lambda_2 = \alpha - j\beta$. One has

$$A(u + jv) = (\alpha + j\beta)(u + jv) \qquad A(u - jv) = (\alpha - j\beta)(u - jv)$$

Summing and subtracting: $Au = \alpha u - \beta v \qquad Av = \beta u + \alpha v$

$$\Rightarrow T = [\mathbf{v}_1 \quad \mathbf{v}_2]$$

$\mathbf{v}_1$ and $\mathbf{v}_2$ are the real eigenvectors of A associated with λ_1 and λ_2, respectively. Defining the change of coordinates

$$r = \sqrt{z_1^2 + z_2^2} \qquad \theta = \tan^{-1}\left(\frac{z_2}{z_1}\right)$$

we can write the dynamic equations in polar coordinates as

$$\dot{r} = \alpha r, \qquad \dot{\theta} = \beta$$

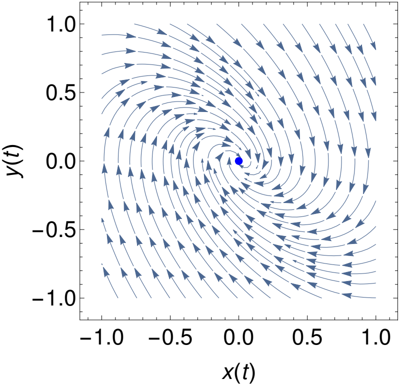
with solution

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t$$

Behavior of Linear second Order Systems

Case 3a: $\alpha < 0$

The origin is called **Stable Focus**



Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned}\dot{x} &= -2.2x - 2.9y \\ \dot{y} &= 2.9x + 2y\end{aligned}$$

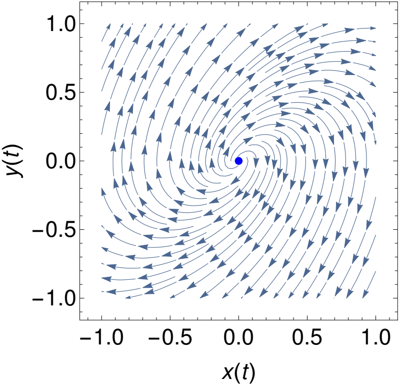
eigenvalues : $\lambda_1 = -0.1 + 2j;\ \lambda_2 = -0.1 - 2j \Longrightarrow$ *Stable Focus*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 2.9 \\ -2.1 - 2j \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 2.9 \\ -2.1 + 2j \end{pmatrix}$

Behavior of Linear second Order Systems

Case 3a: $\alpha > 0$

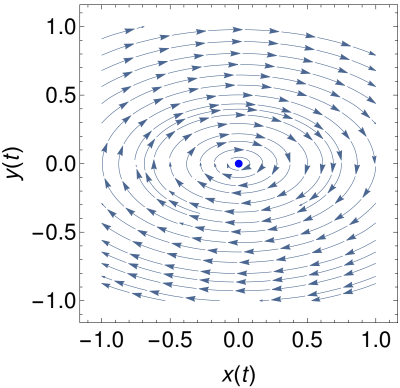
The origin is called **Unstable Focus**

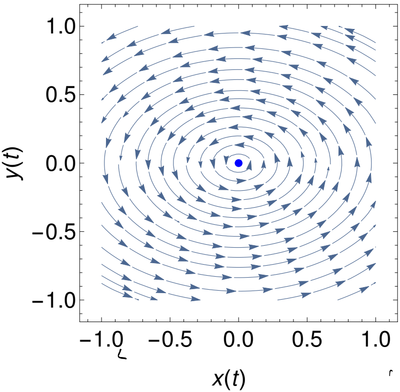


Behavior of Linear second Order Systems

Case 3a: α

The origin is called **Center**





Behavior of Linear second Order Systems

Example: Consider the linear system

$$\begin{aligned} \dot{x} &= -3x + 10y \\ \dot{y} &= -x + 3y \end{aligned}$$

eigenvalues : $\lambda_1 = 0 + j;\ \lambda_2 = 0 - j \Longrightarrow$ *Center*

eigen-vectors : $\overrightarrow{v_1} = \begin{pmatrix} 10 \\ 3 + j \end{pmatrix}, \quad \overrightarrow{v_2} = \begin{pmatrix} 10 \\ 3 - j \end{pmatrix}$

Behavior of Linear second Order Systems

Phase Diagram All of these behaviors can be classified according to the trace T_r and the determinant Det of the matrix A. Recall that for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Find the eigenvalues of A :

$$det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - T_r\lambda + Det = 0$$

Thus the eigenvalues are

$$\begin{array}{lcl} Tr(A) & \equiv & a + d = \lambda_1 + \lambda_2 \Rightarrow \frac{1}{2}T_r(A) & = & m \quad \text{(mean)} \\ Det(A) & \equiv & ad - bc = \lambda_1\lambda_2 & = & p \ (product) \\ \lambda_1, \lambda_2 & = & m \pm \sqrt{m^2 - p} & & \end{array}$$

Behavior of Linear second Order Systems

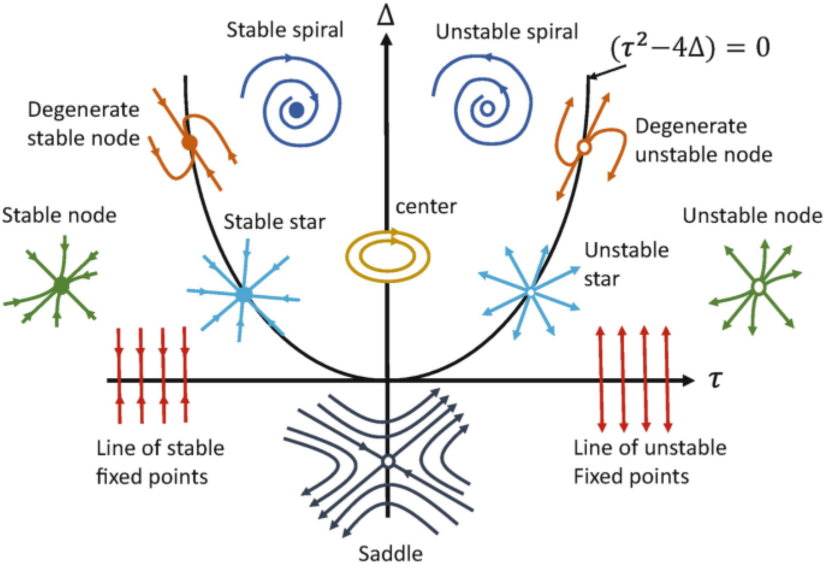
The values of (m, p) determine the equilibrium type.

- If $p < 0$, then the eigenvalues are real with opposite signs (**saddle node**).
- if $m^2 < p$, then the eigenvalues are complex with a real part (**spiral**: unstable if $m > 0$ and stable if $m < 0$).
- If $m = 0$ and $p > 0$, then the eigenvalues are purely imaginary (a **center**).
- $p > 0$ and $m^2 > p$ then the eigenvalues are real with the same sign (a **node**: stable if $m > 0$ and unstable if $m < 0$).

For linear system
The global qualitative behavior is determined by the type of equilibium point.

For nonlinear system Only local qualitative behavior in the vicinty of equilibrium point is determined by the type of equilibium point.

Behavior of Linear second Order Systems



Qualitative Behavior Near Equilibria

Given the nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \quad (4)$$

let us assume $x_e = (x_{1e}, x_{2e})$ is an equilibrium point of (4) i,e.,

$$f_1(x_{1e}, x_{2e}) = f_2(x_{1e}, x_{2e}) = 0$$

f_1, f_2 are continuously differentiable about (x_{1e}, x_{2e})
Since we are interested in trajectories near (x_{1e}, x_{2e}), define

$$x_1 = x_{1e} + \tilde{x}_1, \qquad x_2 = x_{2e} + \tilde{x}_2$$

$\tilde{x}_1$, $\tilde{x}_2$ are small perturbations form equilibrium point.
Expanding (4) into its Taylor series

$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \left.\frac{\delta f_1(x)}{\delta x_1}\right|_{x_e} \tilde{x}_1 + \left.\frac{\delta f_1(x)}{\delta x_2}\right|_{x_e} \tilde{x}_2 + H.O.T$$

$$\dot{x}_1 = \dot{x}_{1e} + \dot{\tilde{x}}_1 = \underbrace{f_1(x_{1e}, x_{2e})}_{0} + \left.\frac{\delta f_1(x)}{\delta x_1}\right|_{x_e} \tilde{x}_1 + \left.\frac{\delta f_1(x)}{\delta x_2}\right|_{x_e} \tilde{x}_2 + H.O.T$$

For sufficiently small neighborhood of equilibrium points, H.O.T. are negligible

$$\begin{array}{rcl} \dot{\tilde{x}}_1 & = & a_{11}\tilde{x}_1 + a_{12}\tilde{x}_2 \\ \dot{\tilde{x}}_2 & = & a_{21}\tilde{x}_1 + a_{22}\tilde{x}_2 \end{array}, \quad a_{i,j} = \left.\frac{\delta f_i}{\delta x}\right|_{x_e}, \quad i = 1, 2.$$

The equilibrium point of the linear system is

$$\tilde{x} = A\tilde{x}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix} = \left.\frac{\delta f}{\delta x}\right|_{x_e}$$

Matrix $\frac{\delta f}{\delta x}$ is called Jacobian Matrix .

The trajectories of the nonlinear system in a small neighborhood of an equilibrium point are close to the trajectories of its linearization about that point:

if the origin of the linearized state equation is a

- stable (unstable) node, or a stable (unstable) focus or a saddle point,

then in a small neighborhood of the equilibrium point, the trajectory of the nonlinear system will behave like a

- stable (unstable) node, or a stable (unstable) focus or a saddle point.

Qualitative Behavior Near Equilibria

Example

$$\begin{aligned} \dot{x}_1 &= 3x_1 - x_1 x_2 \\ \dot{x}_2 &= -4x_2 + x_1 x_2 \end{aligned}$$

Equilibrium points: $f(\bar{x}) = 0 \Longrightarrow \bar{x} = (0,0); \, \bar{x} = (4,3)$

Linearization :

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 3 - x_2 & -x_1 \\ x_2 & -4 + x_1 \end{bmatrix}$$

Linearization around $\bar{x} = (0,0)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(0,0)} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}; \quad \text{Eigenvalues} := \{3, -4\}$$

$\Rightarrow$ **Saddle** type of equilibrium.

Linearization around $\bar{x} = (4,3)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(4,3)} = \begin{bmatrix} 0 & -4 \\ 3 & 0 \end{bmatrix}; \quad \text{Eigenvalues} := \{0 \pm j\sqrt{12}\}$$

$\Rightarrow$ **Center** type of equilibrium.

Qualitative Behavior Near Equilibria

Example

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - 0.2x_2 \end{aligned}$$

Equilibrium points: $f(\bar{x}) = 0 \Longrightarrow \bar{x} = (0,0);\ \bar{x} = (1,0);\ \bar{x} = (-1,0)$

Linearization :

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -0.2 \end{bmatrix}$$

Linearization around $\bar{x} = (0,0)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -0.2 \end{bmatrix}; \quad \text{Eigenvalues} := \{-1.1, 0.9\}$$

$\Rightarrow$ **Saddle** type of equilibrium.

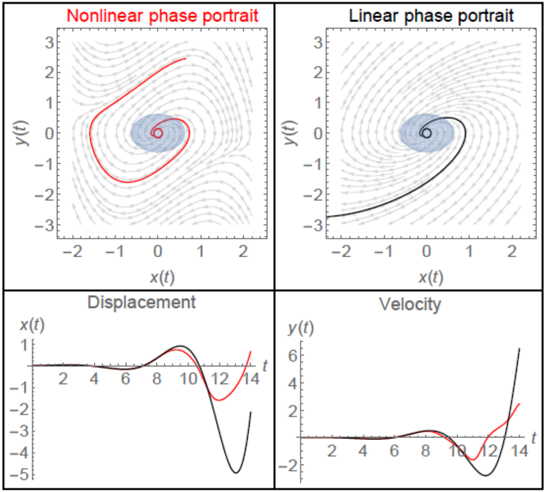
Linearization around $\bar{x} = (1,0)$ and $\bar{x} = (-1,0)$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{\bar{x}=(\pm 1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -0.2 \end{bmatrix}; \quad \text{Eigenvalues} := \{-0.1 \pm \sqrt{2}\}$$

$\Rightarrow$ **Spiral Sink** type of equilibrium.

Qualitative Behavior Near Equilibria

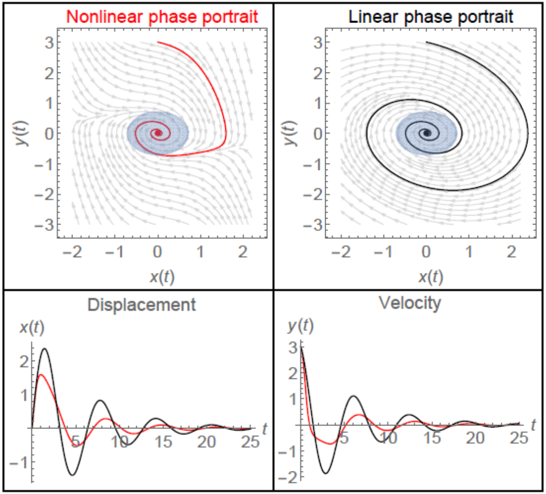
Nonlinear vs. linearised phase portrait



Example: The Liénard equation (red) and its linearization (black). Parameter $\mu = 0.95$

Qualitative Behavior Near Equilibria

Nonlinear vs. linearized phase portrait



Example: The Liénard equation (red) and its linearisation (black). Parameter $\mu = -0.35$

Example: ambiguous borderline case

$$\begin{cases} \dot{x}_1 = -x_2 + \underbrace{\mu x_1(x_1^2 + x_2^2)}_{\text{nonlinear terms}} \\ \dot{x}_2 = x_1 + \overbrace{\mu x_2(x_1^2 + x_2^2)} \end{cases} \tag{5}$$

Fixed point : $(x_{1e}, x_{2e}) = (0, 0)$.
Linearization :

$$J = \begin{pmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{pmatrix} \Bigg|_{0,0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

It is important to note that the linearized system does not depend on the control parameter μ.

Classification of the fixed point of the linearized system.

Trace of the system matrix is $T_r = 0$.

Determinant of the system matrix is $p = 1$.

The linear fixed point is a **centre**.

Example : The Lotka-Volterra competitive cohabitation model2 from ecology competitive cohabitation of rabbits and sheep. The model has the following form:

$$\begin{cases} \dot{x} &= x(3-x) - 2xy \\ \dot{y} &= y(2-y) - xy \end{cases} \tag{6}$$

where x and y are the sizes of rabbit and sheep populations, respectively.