Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Hyperbolic equilibrium point: linearization has no eigenvalues on the imaginary axis.

Hartman-Grobman Theorem:

If x_e is a hyperbolic equilibrium of a planar dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ then there is neighborhood U around x_e and a *homeomorphism*[1]

$$h : U \to \mathbb{R}^2$$

that maps the nonlinear trajectories in U to the linear trajectories in $\mathbb{R}^2$.

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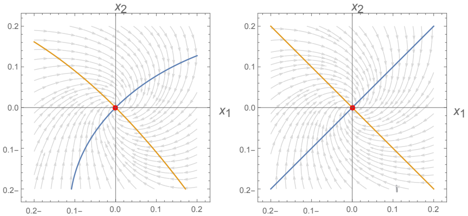
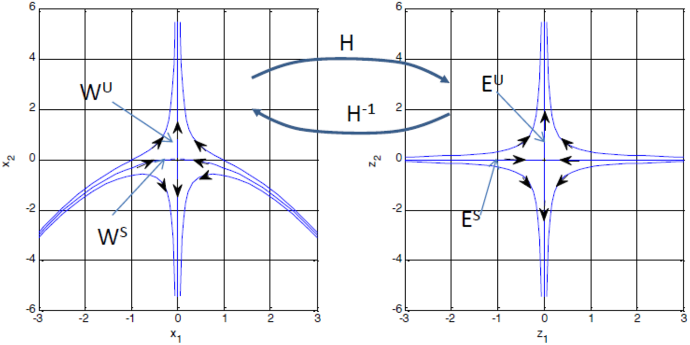
homeomorphism: a continuous map with a continuous inverse (i.e. a change of coordinates)



Example: Consider the non-linear autonomous system

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= x_2 + x_1^2\end{aligned}$$

Equilibrium point : $\dot{x} = 0 \Rightarrow \bar{x} = (0,0)^T$.
Eigenvalues: $\lambda_1 = -1,\ \lambda_2 = 1 \Longrightarrow$ **saddle** type of equilibrium.



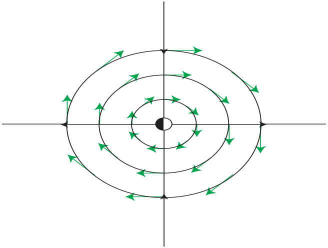
Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

Example

$$\begin{aligned} \dot{x}_1 &= -x_2 + \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 + \mu x_2(x_1^2 + x_2^2) \end{aligned} \qquad (7)$$

There is only one equilibium point at (0,0), and the linearized system at this point is

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Longrightarrow \lambda_{1,2} = \{\pm J\}$$



$\Longrightarrow$ the equilibrium point is **center**. Since this equilibrium point is *non-hyperbolic* $\Longrightarrow$ No conclusion about the behavior of the nonlinear system near (0,0)

Frame Title

System (7) is analysed in polar coordinates. Usually a direct coordinate transform

$$\left\{\begin{array}{lcl} x_1 & = & r\cos\theta \\ x_2 & = & r\sin\theta \end{array}\right. , \left\{\begin{array}{lcl} r & = & \sqrt{x_1^2+x_2^2} \\ \theta & = & \tan\left(x_2/x_1\right) \end{array}\right. \tag{9}$$

where $r = r(t)$ and $\theta = \theta(t)$, is used. We are searching a system in the form:

$$\left\{\begin{array}{lcl} \dot{r} & = & f_1(r,\theta) \\ \dot{\theta} & = & f_2(r,\theta) \end{array}\right. \tag{10}$$

where functions $f_1(r,\theta)$ and $f_2(r,\theta)$ are to be determined. We are interested in temporal dynamics of (9)

$$\left\{\begin{array}{lll} 2r\dot{r} & = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 & \Rightarrow \dot{r} = \dfrac{x_1 f_1(x_1,x_2) + x_2 f_2(x_1,x_2)}{r} \\ \sec^2\theta\dot{\theta} & = (1+\tan^2\theta)\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2} & \Rightarrow \dot{\theta} = \dfrac{x_1 f_2(x_1,x_2) - x_2 f_1(x_1,x_2)}{x_1^2+x_2^2} \end{array}\right.$$

Phase Portraits of Nonlinear Systems Near Hyperbolic Equilibria

After developing (11). The system (7) has been represented in polar coordinates. Resulting decoupled equations

$$\begin{array}{rcrcl} \dot{x}_1 & = & -x_2 & + & \mu x_1(x_1^2 + x_2^2) \\ \dot{x}_2 & = & x_1 & + & \mu x_2(x_1^2 + x_2^2) \end{array} \Longrightarrow \left\{ \begin{array}{rcl} \dot{r} & = & \mu r^3 \\ \dot{\theta} & = & 1 \end{array} \right.$$

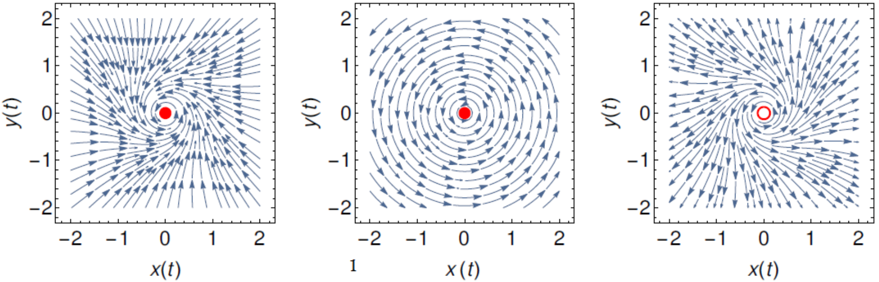


Figure: $\mu < 0$, $\mu = 0$, $\mu > 0$

Non-existence of Periodic Orbits

Bendixson criterion gives a sufficient condition for detecting the absence of periodic orbits for second-order systems (Limit cycles or neutrally stable cycles).

Bendixson criterion:

For a time-invariant planar system

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2)$$

If $div(f) = \nabla . f(x) = [\partial/\partial x_1 \quad \partial/\partial x_2] \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign in a simply connected region D, then there are no periodic orbits lying entirely in D.

Example 1: $\dot{x} = Ax, x \in \mathbb{R}^2$ can have periodic orbits only if div f= trace(A)=0.,

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}$$

Unless $trace(A) = 0 \Longrightarrow$ non periodic orbits.

Example 2:

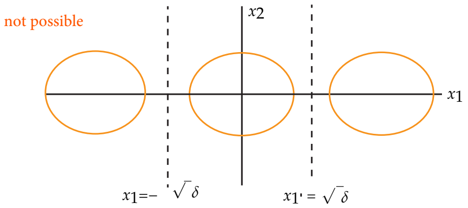
$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2, \qquad \delta > 0\end{aligned}$$

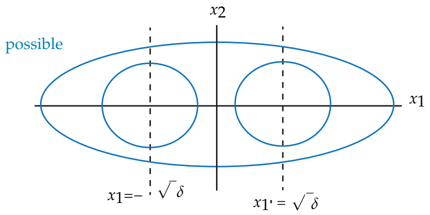
$$\nabla . f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = x_1^2 - \delta$$

$$\nabla . f(x) = 0, \quad \text{then} \quad x_1 = \pm\sqrt{\delta}$$

Therefore, no periodic orbit can lie entirely in the region

$$x_1 \in] -\infty, -\sqrt{\delta}[, \quad] -\sqrt{\delta}, \sqrt{\delta}[, \quad]\sqrt{\delta}, +\infty[$$





Periodic Orbits in the Plane

Let $\phi(t, x_0)$ denotes the solution of $\dot{x} = f(x)$ with initial condition $x(0) = x_0$.

Definition: Invariant sets

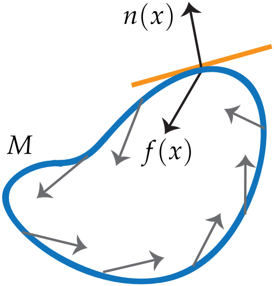
A set $M \subseteq \mathbb{R}^n$ is **positively invariant** if, for each $x_0 \in M$, $\phi(t, x_0) \in M$ for all $t \geq 0$.

Theorem

If $V : \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^1$ and $M = \{x : V(x) \leq c\}$, then M is invariant if

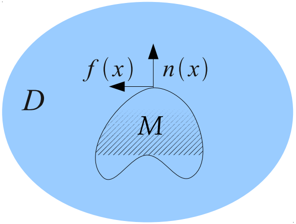
$$f(x).\nabla V(x) \leq 0 \quad \forall x : V(x) = c$$

i.e. if x is on the boundary of M, then the vector $f(x)$ points into M.

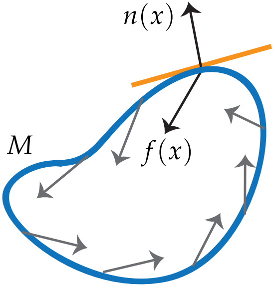


Non-existence of Periodic Orbits

Example: Consider a closed orbit $\dot{x} = f(x)$.
$f(x)$ is tangential to the trajectory x. Along this closed trajectory : $f^T(x).\vec{n} = 0$.
The interior of any closed trajectory is a positively invariant set.



Example:
If $f(x)^T.\vec{n} \leq 0$ then M is positively invariant.
$\vec{n}$: outward normal on a boundary of M.
Along boundary of M:for all $x \in \partial M \Rightarrow [f(x)]^T.\vec{n} \leq 0$.



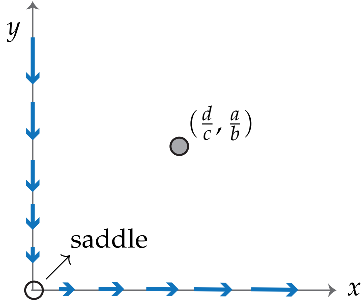
Periodic Orbits in the Plane

Example: Predator-prey model

$$\begin{aligned} \text{prey :} \quad & \dot{x}_1 &= (a - bx_2)x_1 \\ \text{predator :} \quad & \dot{x}_1 &= (cx_1 - d)x_2 \end{aligned}$$

a,b,c,d positive parameters.

Equilibrium points : $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} d/c \\ a/b \end{bmatrix}.$



Clearly $[f(x)]^T.\vec{n} = 0$ along the boundary of $M = \{x_1 \geq 0, x_2 \geq 0\}$, which means the first quadrant M is positively invariant.

Linearization around $\bar{x} = (0,0)$: $A = \frac{\partial f}{\partial x}\Big|_{\bar{x}} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix}$.

$\lambda_1 = a > 0$ and $\lambda_2 = -d < 0 \Longrightarrow$ **saddle** type of equilibrium.

Periodic Orbits in the Plane

Example 2:

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{aligned}$$

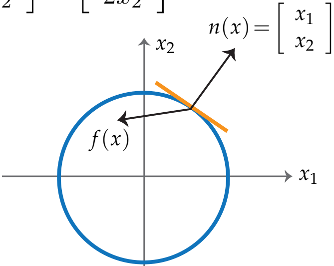
Show that $B_r := \{x \in \mathbb{R}^2 / x_1^2 + x_2 \leq r^2\}$ is positively invariant for sufficiently large r (to be determined). We want to calculate $[f(x)]^T n(x)$

$$V(x) = x_1^2 + x_2^2 = r^2 \Rightarrow n(x) = \nabla V(x) = \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\begin{aligned}[f(x)]^T.n(x) &= = f_1\frac{\partial V}{\partial x_1} + f_2\frac{\partial V}{\partial x_2}\\ &= -2(x_1^2 + x_2^2)^2 + 2x_1^2 + 2x_2^2 - 2x_1x_2\end{aligned}$$

$-2x_2x_2 \leq x_1^2 + x_2^2$ (completion of squares).

Therefore $[f(x)]^T.n(x) \leq -2r^2(r^2 - 3/2)) \leq 0$ if $r^2 \geq 3/2$.



Existence Theorem of Limit Cycle

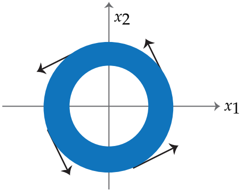
Poincaré-Bendixson Theorem:

Let M be a compact (closed and bounded) set in $\mathbb{R}^2$, which is positively invariant for $\dot{x} = f(x)$, $x \in \mathbb{R}^2$. If M does not contain an equilibrium point, then it contains a periodic orbit.

The "no equilibrium condition" in PB Theorem can be relaxed as: " M can have one equilibrium point which is either an unstable focus or an unstable node, then there is a periodic orbit.

Example: harmonic oscillator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



For any $R > r > 0$, the ring $\{x : r^2 \le x_2 + x_2 \le R^2\}$ is compact, invariant and contain no equilibria.

Existence Theorem of Limit Cycle

$[f(x)]^T.n(x) = 0$ everywhere and PB Theorem state there exists a periodic orbit (or more) in M.
Example:2

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1 x_2^2 \\ \dot{x}_1 &= -x_1 + x_1^2 x_2 \end{aligned}$$

Linearization around the equilibrium at $\bar{x} = (0\ 0)^T$ yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which exhibits a continuum of periodic solutions. However, for this nonlinear system, we have

$$\nabla.f(x) = x_1^2 + x_2^2 > 0, \quad \forall x \neq 0$$

Hence, Bendixson theorem leads to the conclusion that this dynamical system has no nontrivial periodic solutions.

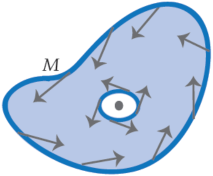
Existence Theorem of Limit Cycle

Example:3

$$\begin{array}{rcr}\dot{x}_1 & = & x_1 + x_2 - x_1(x_1^2 + x_2^2)\\ \dot{x}_2 & = & -2x_1 + x_2 - x_2(x_1^2 + x_2^2)\end{array}$$

$[f(x)]^T.n(x) \leq 0$ iff $r^2 = x_1^2 + x_2^2 > 3/2$. i.e. $B_r = x \in \mathbb{R}^2/x_1^2 + x_2^2 \leq r^2$ is positively invariant $r \geq \sqrt{3/2}$ but contains the equilibrium $x_e = 0$.

$$\left.\frac{\partial f}{\partial x}\right|_{x_e=0} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix},\ \lambda_{1,2} = 1 \pm j\sqrt{2},\ \text{unstable focus}$$



Therefore, B_r must contain a periodic orbit.

Limit Cycle: Stable Limit Cycle

All trajectories in the vicinity of the limit cycle converges to it as $t \to \infty$

Example:

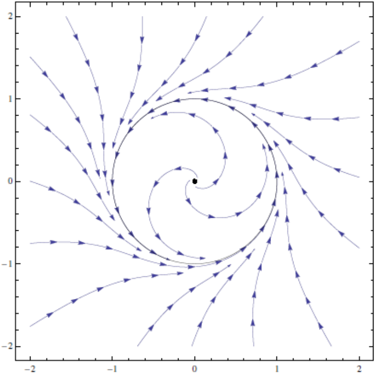
$$\begin{aligned} \dot{x}_1 &= \quad x_2 - x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 &= -x_1 - x_2(x_1^2 + x_2^2 - 1) \end{aligned}$$

$$\Longrightarrow \begin{cases} \dot{r} &= -r(r^2 - 1) \\ \dot{\theta} &= -1 \end{cases} \qquad (12)$$

if $r > 1 \to \dot{r} > 0$ converging

if $r > 1 \to r < 0$ converging

if $r = 1 \to \dot{r} = 0$ remaining



Limit Cycle: Unstable Limit Cycle

All trajectories in the vicinity of the limit cycle diverges from it as $t \rightarrow$

Example

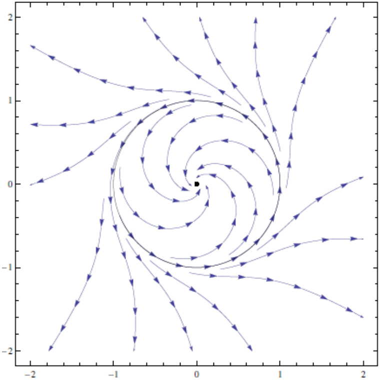
$$\begin{array}{rcl} \dot{x}_1 & = & x_2 + x_1(x_1^2 + x_2^2 - 1) \\ \dot{x}_1 & = & -x_1 + x_2(x_1^2 + x_2^2 - 1) \end{array}$$

$$\Longrightarrow \left\{ \begin{array}{rcl} \dot{r} & = & r(r^2 - 1) \\ \dot{\theta} & = & -1 \end{array} \right. \qquad (13)$$

if $r < 1 \rightarrow \dot{r} < 0$ diverging

if $r > 1 \rightarrow r > 0$ diverging

if $r = 1 \rightarrow \dot{r} = 0$ remaining



Limit Cycle: half-stable Limit Cycle

Some of the trajectories in the vicinity of the limit cycle converges to it, while others diverge from it as $t \to \infty$

Example:

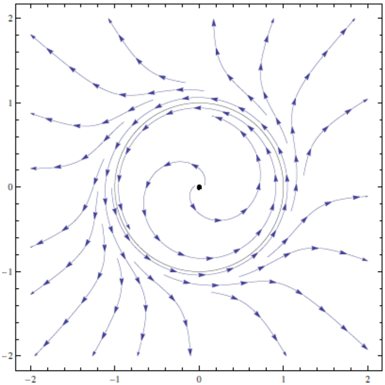
$$\begin{aligned} \dot{x}_1 &= x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \\ \dot{x}_1 &= -x_1 - x_2(x_1^2 + x_2^2 - 1)^2 \end{aligned}$$

$$\Longrightarrow \left\{ \begin{aligned} \dot{r} &= -r(r^2 - 1)^2 \\ \dot{\theta} &= -1 \end{aligned} \right. \qquad (14)$$

if $r < 1 \to \dot{r} < 0$ diverging

if $r > 1 \to r < 0$ converging

if $r = 1 \to \dot{r} = 0$ remaining



Bifurcation: Holf Bifurcation

Example: Supercritical Hopf bifurcation

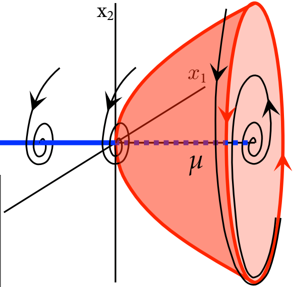
$$\left\{\begin{array}{lcl}\dot{x}_1 & = & -x_2 + x_1(\mu - x_1^2 - x_2^2)\\ \dot{x}_2 & = & +x_1 - x_2(\mu - x_1^2 - x_2^2)\end{array}\right. \Longrightarrow \left\{\begin{array}{lcl}\dot{r} & = & r(\mu - r^2)\\ \dot{\theta} & = & 1\end{array}\right. \tag{15}$$

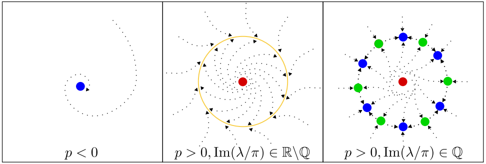
Equilibrium points : $r(\mu - \bar{r}^2) = 0.$
Note that a positive equilibrium for the r subsystem means a limit cycle in the $(x1, x2)$ plane.

$\mu < 0$: stable equilibrium at $r = 0$.

$\mu > 0$ unstable equilibrium point ar $r = 0$ and stable limit cycle at $r = \sqrt{\mu}$.

The origin loses stability at $\mu = 0$ and a stable limit cycle emerges.





Bifurcation: Holf Bifurcation

In Supercritical Hopf bifurcation by increase of μ near zero, the stable equilibrium point becomes unstable but a stable limit cycle appears. Hence, this is a Safe bifurcation.