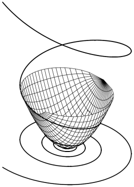
Stability



Stability theory is divided into three parts:

- Stability of equilibrium points.
- Stability of periodic orbits
- Input/output stability

Alexander Mikhailovich Lyapunov (1957-1918)

- Russian mathematician and physicist.
- Known for his development of the stability theory of dynamical systems.

If the total energy is dissipated, then the system must be stable.

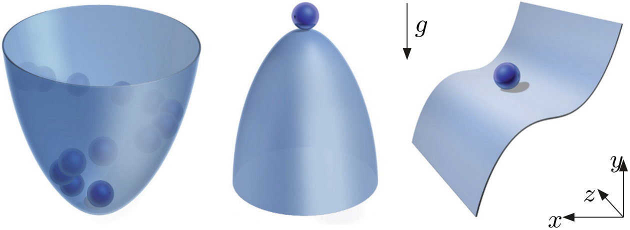
Stability of Autonomous Systems

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a *locally Lipschitz* map from a domain $\mathcal{D} \subset$ into $\mathbb{R}^n$. Suppose that the system (1) has an equilibrium point $\bar{x} \in \mathcal{D}$, i.e., $f(\bar{x}) = 0$.

> Without loss of generality and the simplicity of notation, we assume the equilibrium is located at the origin.



Autonomous Systems : stability of an equilibrium state

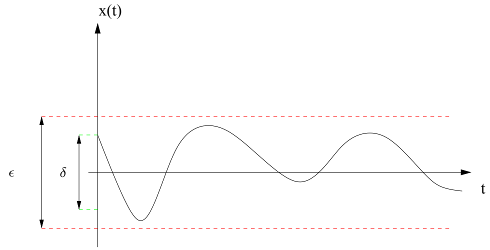
Stability in the sense of Lyapunov

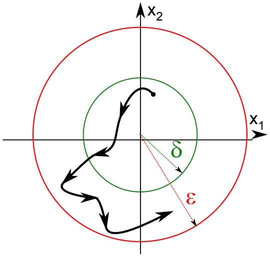
The equilibrium point $\bar{x} = 0$ of $\dot{x} = f(x)$ is

stable, if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \epsilon, \forall t \geq 0.$$

unstable, if it is not stable.





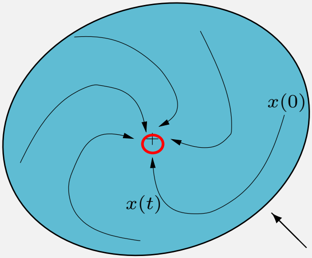
Autonomous Systems : stability of an equilibrium state

"Stability is a property of the equilibrium, not of the system"

Stability of the equilibrium is equivalent to stability of the system only when there exists only one equilibrium (e.g., linear systems). In this case **stability** $\equiv$ **global stability.**

The region of **attraction** of the equilibrium point $\bar{x} = 0$ of (1) is is the set of all initial conditions $x(0)$ for which

$$x(t) \to 0 \quad \text{as} \quad t \to \infty.$$



Autonomous Systems : stability of an equilibrium state

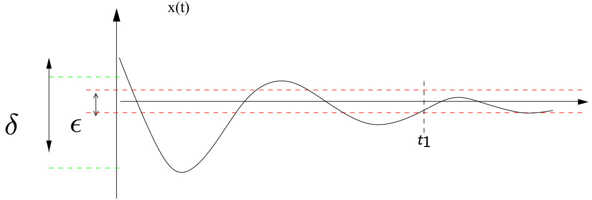
Stability in the sense of Lyapunov

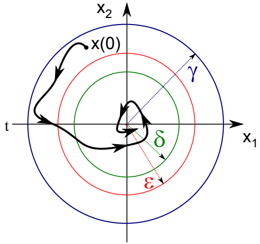
The equilibrium point $\bar{x} = 0$ of (1) is

attractive, if there exist δ such that :

$$\|x(0)\| < \delta \Longrightarrow \lim_{t\to\infty} x(t) = 0.$$

local asymptotically stable (a.s) if it is **stable** and **attractive**





Autonomous Systems : stability of an equilibrium state

The equilibrium point $\bar{x} = 0$ of (1) is

- **exponentially stable** (e.s), if there exists $\alpha >, \beta > 0$ and $\delta > 0$ such that
$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$
- **globally asymptotically stable** (g.a.s) if it is stable and globally attractive, i.e $\lim_{t\to\infty} x(t) = 0$. for all $x(0) \in \mathbb{R}^n$.
- **globally exponentially stable**(g.e.s), for all $x(0) \in \mathbb{R}^n$, there exists $\beta > 0$ and $\alpha > 0$ such that
$$\|x(t)\| \leq \beta \|x(0)\| e^{-\alpha t}, \forall t \geq 0$$

exponential stability is a special case of asymptotic stability.

Stability, AS, ES: local concepts (for x(0) sufficiently close to ")