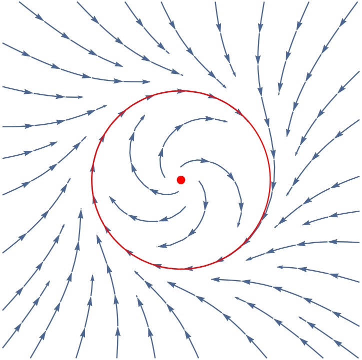
Remarks on stability

Attractively does not imply asymptotic stability

Example: Consider the second-order system with state variables x_1 and x_2 whose dynamics are most easily described in polar coordinates via the equations

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\theta} &= \sin^2(\theta/2) \end{aligned} \qquad (2)$$



where $r = \sqrt{x_1^2 + x_2^2}$ and $\theta = \arctan(x_2/x_1)$, $\theta \in [0, 2\pi[$.

This system has two equilibrium points : (r^*, θ^*)=(0,0) and (r^*, θ^*)=(1,0).

The fixed point at zero is clearly unstable.

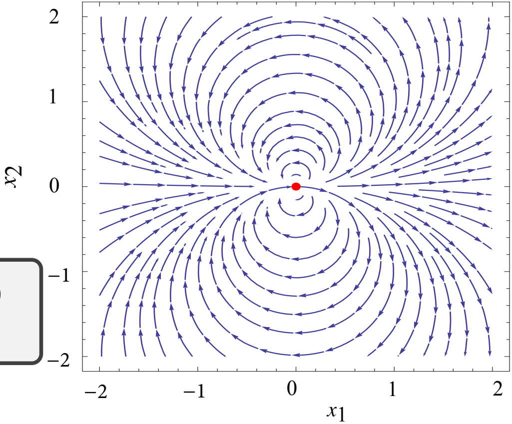
The fixed point with $r^* = 1$ attracts all other trajectories, but it is not stable by any of our definitions.

Remarks on stability : Chaotic attractor

Example

$$\begin{aligned}\dot{x}_1 &= x_1^2 - x_2^2\\ \dot{x}_2 &= 2x_1x_2\end{aligned}$$

All trajectories converge to $x_e = 0$ but x_e is not stable.



Convergence by itself does not imply Stability

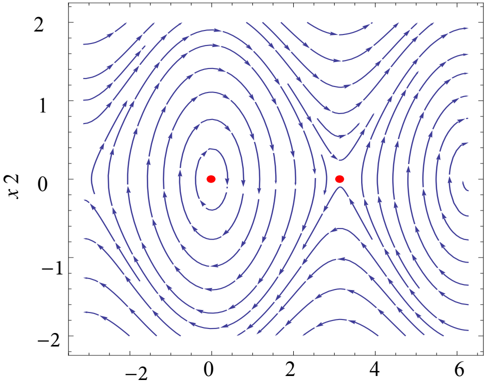
Remarks on stability

There exist Lyapunov-stable sets that are not attractors.

Example: stable system but not attractor

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) \end{aligned}$$

Show that an equilibrium point $x^- = [2k\pi, 0]^T$ is stable but not .attractor



Remarks on stability

Example:

$$\dot{x} = -x^2 \Longrightarrow x(t) = \frac{x(0)}{1 + tx(0)}$$

$\bar{x} = 0$ is GAS (but not GES).

Problems

How to check the stability properties of $\bar{x} = 0$ WITHOUT computing state trajectories ?

if $\bar{x} = 0$ is A.S how to compute a region of attraction ?

The analysis of the linearized system around $\bar{x} = 0$ MIGHT allow one to check local stability. How to proceeed when no conclusion can be drawn using the linearized system ?

Need of a more complete approach: **Lyapunov direct method**

Remarks on stability

Example

$$\dot{x} = ax - x^5$$

$\bar{x} = 0$ is an equilibrium state.
Linearized system: $\dot{x} = ax$

$a < 0 \Longrightarrow \bar{x} = 0$ is A.S

$a > 0 \Longrightarrow \bar{x} = 0$ is unstable

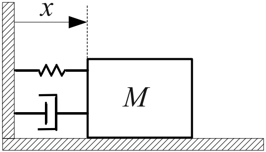
a=0 ?

For $a = 0$ one has $\dot{x} = -x^5$ and with the Lyapunov direct method one can show that $\bar{x} = 0$ is AS.

One way to check **stability** is to **plot** trajectories and see what is going on. This approach has limited utility because it requires solution to differential equation (which may difficult to solve!)

Lyapunov direct method

Example



Model

$$M\ddot{x} = \underbrace{-b\dot{x}|\dot{x}|}_{\text{NL damping}} - \underbrace{\left(k_0 x + k_1 x^3\right)}_{\text{NL elasticforce}}$$

Defining $x_1 = x$ and $x_2 = \dot{x}_1$

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{M} x_2 |x_2| - \frac{k_0}{M} x_1 - \frac{k_1}{M} x_1^3 \end{array} \right. \Rightarrow \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{is stable/AS/ES}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} - 3\frac{k_1}{M} x_1^2 & -\frac{2b}{M} x_2 \mathrm{sgn}(x_2) \end{bmatrix} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{k_0}{M} & 0 \end{bmatrix}$$

Eigenvalues: $\lambda_{1,2} = \pm j\sqrt{k_0/M}$.

No conclusion on $\bar{x} = 0$ using the linearized system.

Lyapunov direct method

Consider the total energy of the system:

$$V(x) = \underbrace{\frac{1}{2}Mx_2^2}_{kinetic} + \underbrace{\frac{1}{2}k_0x^2 + \frac{1}{4}k_1x_1^4}_{potential}$$

Remark: zero energy⇔x$_1 = x_2 = 0$ (equilibrium state)

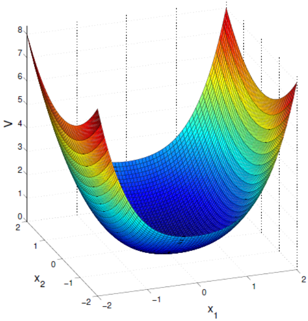
Instantaneous energy change:

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x}.\frac{dx}{dt} = \left[\frac{\partial V}{\partial x_1}\ \ \frac{\partial V}{\partial x_2}\right]\left[\begin{array}{c}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\end{array}\right] = \left(k_0x_1 + k_1x_1^3\right)\dot{x}_1 + Mx_2\dot{x}_2\\ &= (k_0x_1 + k_1x_1^3)x_2 + Mx_2(-\frac{b}{M}x_2|x_2| - \frac{k_0}{M}x_1 - \frac{k_1}{M}x_1^3) = -bx_2^2|x_2|\end{aligned}$$

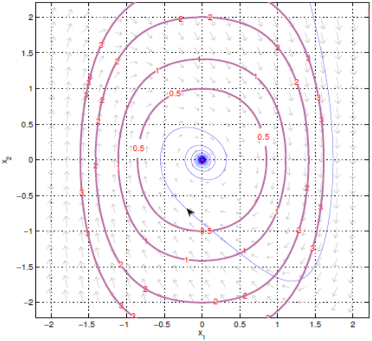
$-bx_2^2|x_2| \leq 0$ independently of $x(0)$ ⇒ the energy can only decrease with time independently of $x(0)$

Lyapunov direct method: a first example

Energy V(x)



Phase plane



Energy is a "measure" of the distance of x from the origin

- if it can only decrease, then $\bar{x} = 0$ should be stable.

Lyapunov direct method is based on energy-like functions $V(x)$ and the analysis of the function

Derivative along the trajectory

Definition: Lyapunov Function

Let $V : \mathcal{D} \to \mathbb{R}$ be a *continuously differentiable* function defined in a domain $D \in \mathbb{R}^n$ that contains the origin. The derivative of V along the trajectory (solution) of $\dot{x} = f(x)$ denoted by $\dot{V}(x)$ is given by

$$\begin{aligned}\dot{V}(x) = \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} = \nabla V . f(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \cdots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}\end{aligned}$$

Lyapunov stability theory

Definition: Positive Definite Functions

A function $V : \mathcal{D} \to \mathbb{R}$ is *positive semi definite* in $\mathcal{D}$ if

(i). $V(x) = 0$ if and only if $x = 0$.

(ii). $V(x) \geq 0, \forall x$in $\mathcal{D} - \{0\}$.

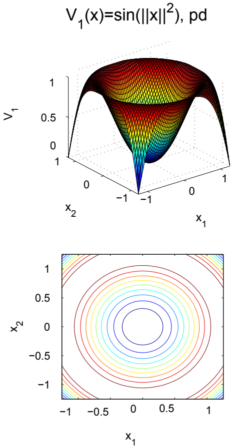
A function $V : \mathcal{D} \to \mathbb{R}$ is *positive definite* in $\mathcal{D}$ if

(ii'). $V(x) > 0, \forall x$in $\mathcal{D} - \{0\}$.

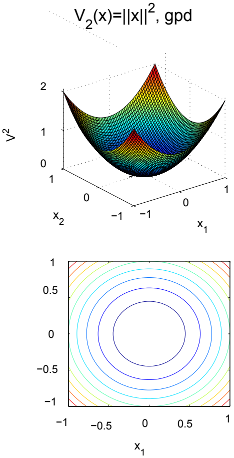
A function $V : \mathcal{D} \to \mathbb{R}$ is *negative definite* in $\mathcal{D}$ if $-V$ is positive definite (semi definite).

Lyapunov stability theory

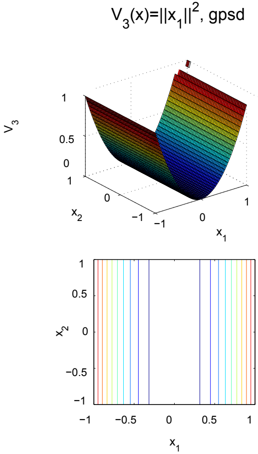
Examples of positive definite functions



$V_1(x)$ pd



$V_2(x)$ gpd



$V_3(x)$ gpsd

Lyapunov Stability Theorem

Theorem

1. Let $\bar{x} = 0$ be an equilibrium for $\dot{x} = f(x)$ and $\mathcal{D} \in \mathbb{R}^n$ be a domain containing $\bar{x} = 0$. If there exists a *continuously differentiable* function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \forall x \in \mathcal{D} - \{0\} \quad \text{(positive definite)}$$

and

$$\dot{V}(x) := \nabla V^T(x) f(x) \leq 0 \quad \forall x \in \text{in } \mathcal{D} \quad \text{(negative semidefinite)}$$

then, $\bar{x} = 0$ is **stable**.

2. If $\dot{V}(x) < 0$, $\forall x \in \mathcal{D} - \{0\}$ (negative definite)

then, $\bar{x} = 0$ is **asymptotically stable**.

3. If, in addition, $\mathcal{D} = \mathbb{R}^n$, and

$$\|x\| \to \infty \Longrightarrow V(x) \to \infty. \quad \text{(radially unbounded)}$$

then $\bar{x} = 0$ is **globally asymptotically stable.**

Lyapunov Stability Theorem

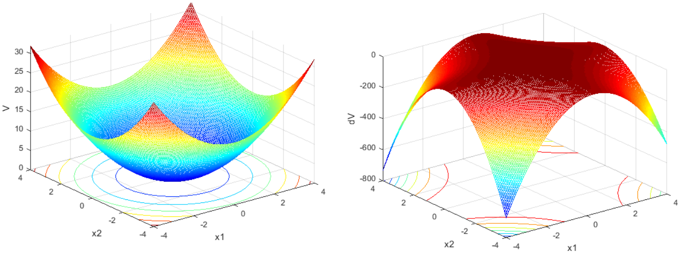
Example

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2\\ \dot{x}_2 &= x_1 - x_2 - x_1^2 x_2\end{aligned}$$

Study the stability of the equilibrium state $\bar{x} = 0$.
Take the Lyapunov function $V(x) = x_1^2 + x_2^2$ (positive definite in $\mathbb{R}^2$)

$$\begin{aligned}\dot{V}(x) &= \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = 2x_1(-x_1 + x_2^2) + 2x_2(x_1 - x_2 - x_1^2 x_2)\\ &= -(x_1 - x_2)^2 - x_2^2(1 - x_1)^2 - x_1^2(1 + x_2^2)\end{aligned}$$

$\dot{V}(x)$ is negative definite $\Longrightarrow$ the system is stable



Lyapunov Stability Theorem

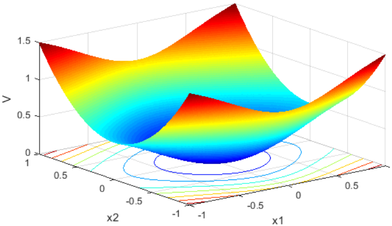
Q.why is radially unboundedness required for G.A.S ?

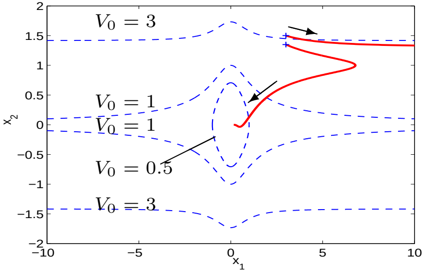
Example: Consider the following *positive definite function*

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

if $x = (x_1, 0)$ then $\|x\| \to \infty \quad as \quad \mathrm{x}_1 \to \infty$.
But $V(x)$ will approach 26 (not radially unbounded)

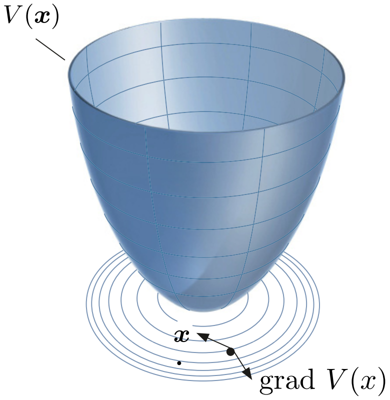
$$\lim_{x_1\to\infty} V(x) = \lim_{x_1\to\infty} \frac{1+x_1^2}{x_1^2} + (5)^2 = 26 \neq +\infty$$





Conservation and Dissipation

Conservation of energy: $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) = 0$, i.e., the vector field $f(x)$ is everywhere orthogonal to the normal $\frac{\partial V}{\partial x}$ to the level surface $V(x) = c$.



Dissipation of energy: $\dot{V}(x) = \dfrac{\partial V}{\partial x} f(x) \leq 0$, i.e., the vector field $f(x)$ and the normal $\frac{\partial V}{\partial x}$ to the level surface $(V(x) = c)$ make an obtuse angle.

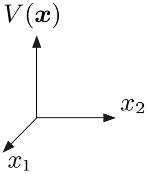


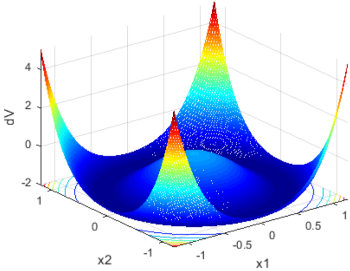
Illustration of the equation $\dot{V}(x) = \dot{x}^T \text{grad } (V(x)) < 0$

Lyapunov Stability Theorem

Not necessary to compute state trajectories: it is enough to check the sign of V and $\dot{V}$ in a neighborhood of the origin.

Example:

$$\begin{aligned}\dot{x}_1 &= x_1\left(x_1^2+x_2^2-2\right)-4x_1x_2^2\\ \dot{x}_2 &= x_2\left(x_1^2+x_2^2-2\right)+4x_1^2x_2\end{aligned}$$



Study the stability of the equilibrium state $\bar{x}=0$
Candidate Lyapunov function: $V(x)=x_1^2+x_2^2$
(positive definite in $\mathbb{R}^2$)

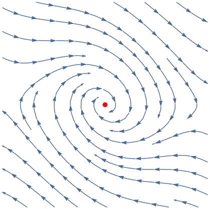
$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}f_1(x)+\frac{\partial V}{\partial x_2}f_2(x)\\ &= 2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-2\right)\end{aligned}$$

In the set level $x_1^2+x_2^2-2<0$ one has $\dot{V}$ is negative definite, therefore $\bar{x}=0$ is AS in the ball centered $B_{\sqrt{2}}(0)$.

Lyapunov Stability Theorem

The choice of the Lyapunov function is not unique.

Example 2: damped pendolum



$$\begin{array}{rcl}\dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_2 - \sin(x_1)\end{array}$$

Study the stability of $\bar{x} = [0 \quad 0]^T$
Lyapunov candidate function: $V(x) = \underbrace{(1 - \cos(x_1))}_{\text{potential en.}} + \underbrace{x_2^2/2}_{\text{kinetic en.}}$

$$\dot{V} = \frac{\partial V}{\partial x_1} f_1(x) + \frac{\partial V}{\partial x_2} f_2(x) = \sin(x_1)x_2 + x_2(-x_2 - \sin(x_1)) = -x_2^2$$

$\dot{V}$ is negative semi definite in $\mathbb{R}^2$ (and then in $B_{2\pi(0)}$) $\implies \bar{x} = 0$ is stable.

Physical intuition tells us the equilibrium is AS but the chosen Lyapunov function certifies only stability

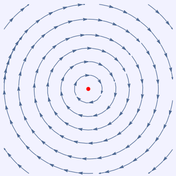
Lyapunov instability theorem

Example Study the stability of $\bar{x} = 0$.

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_1\left(x_1^2 + x_2^4\right) \\ \dot{x}_2 &= -2x_1 + x_2\left(x_1 + x_2^4\right) \end{aligned}$$

Linearized system around $\bar{x} = 0$

$$\begin{aligned} \dot{x}_1 &= 2x_2 \\ \dot{x}_2 &= -2x_1 \end{aligned} \qquad \text{eigenvalues}: \ \pm 2j$$



No conclusion on stability of (3).

Candidate Lyapunov function: $V(x) = (x_1^2 + x_2^2)/2$

$$\begin{aligned} \dot{V} &= x_2\left(2x_1 + x_1\left(x_1^2 + x_2^4\right)\right) + x_2\left(-2x_1 + x_2\left(x_1 + x_2^4\right)\right) \\ &= (x_1^2 + x_2^2)(x_1^2 + x_2^4) \rightarrow \text{positive definite in } \mathbb{R}^2 \end{aligned}$$

Then $\bar{x} = 0$ is unstable

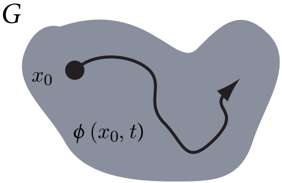
Invariance Principle: Krasovskii-Lasalle LaSalle's Theorem

LaSalle's invariance principle is a tool for assessing **asymptotic stability** properties of $\bar{x} = 0$ for $\dot{x} = f(x)$ when $\dot{V}(x)$ is only semi-definite

Review

A set $G \subseteq \mathbb{R}^n$ is (positively) invariant for $\dot{x} = f(x)$ if

$$x_0 \in G \Longrightarrow \phi(t, x_0) \in G, \quad \forall t \geq 0.$$



Invariance Principle: Local LaSalle theorem

Theorem: LaSalle's Invariance Principle

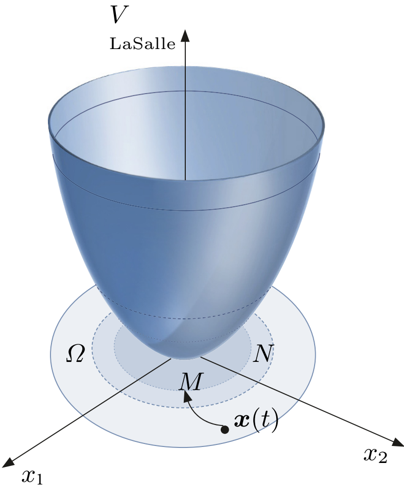
Let $\dot{x} = f(x)$ be a be a system with a compact positively invariant set Ω and let $V_{\text{LaSalle}}(x)$ be a continuously differentiable function with

$$\dot{V}_{\text{LaSalle}} \leq 0$$

for all $x \in \Omega$. Further, let N denote the set of all points $x \in \Omega$ with

$$\dot{V}_{\text{LaSalle}} = 0$$

and let M denote the largest invariant set in N. In this case all solutions $x(t)$ that start within Ω tend to the set M for $t \to \infty$



Invariance Principle: Local LaSalle theorem

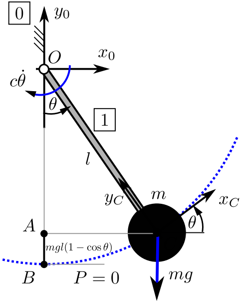
Remarks

- The theorem provides sufficient conditions for Ω to be a region of attraction for the set M
- Notable case: when $M = \{0\}$ the theorem gives a region of attraction (asymptotic stability) for the equilibrium state $\bar{x} = 0$.

Invariance Principle: Local LaSalle theorem

Example: Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 \end{cases}$$



Consider the Lyapunov function

$$V(x) = \underbrace{\frac{g}{l}(1-\cos x_1)}_{\text{potential en.}} + \underbrace{\frac{1}{2}x_2^2}_{\text{kinetic en.}}$$

V(0)=0

V(x)>0 if $x_1 \in (-2\pi, 2\pi)$

Then V is positive definite in $B_2\pi(0)$

$$\dot{V}(x) = \frac{g}{l}\sin x_1 \dot{x}_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \leq 0$$

Then $S := \{(x_1, x_2) | x_2 = 0\}$, i.e., x_1 can be anything and S is the x_1-axis.

Invariance Principle: Local LaSalle theorem

Q. what is the largest invariant set ?

1. $x_2 \equiv 0 \Longrightarrow \dot{x}_2 \equiv 0$
2. $\dot{x}_2 = 0 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}.0 = -\frac{g}{l}\sin(x_1)$

$\sin(x_1) = 0$ must be satisfied. Locally, on the set of $x_1 \in (-\pi, \pi)$, this is only satisfied for $x_1 = 0$. Thus $\bar{x} = 0$ is locally asymptotically stable