Lyapunov theory for LTI systems

Let $x = \bar{x}$ be an equilibrium for the autonomous nonlinear system

$$\dot{x} = f(x) \tag{6}$$

where $f : \mathcal{D} \rightarrow \mathbb{R}^n$ is a continuously differentiable function and $\mathcal{D}$ is a neighborhood of $\bar{x}$. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=\bar{x}}$$

Then:

- $\bar{x}$ is **aymptotically stable** if $\text{Re}(\lambda_i)<0$ for all eigenvalues of A.
- $\bar{x}$ is **stable** if $\text{Re}(\lambda_i)<0$ and $\text{Re}(\lambda_i)=0$ for one of the eigenvalues of A.
- $\bar{x}$ is **unstable** if $\text{Re}(\lambda_i)>0$ for one or more of the eigenvalues of A.
- In linear systems, *local stability* $\Longleftrightarrow$ *global stability.*
- In nonlinear systems, this is not true.

Lyapunov theory for LTI systems

Review: Positive Definite Matrices

Symmetric matrix $M = M^T$ is

1. **positive definite** (pd) if $x^T Mx > 0, \forall x \neq 0.$
2. **positive semi-definite** (psd) if $x^T Mx \geq 0, \forall x \in \mathbb{R}^n.$

Lemma:

$$M = M^T > 0 \iff \lambda_i(M) > 0$$
$$M = M^T \geq 0 \iff \lambda_i(M) \geq 0$$

Properties of the quadratic function $x^T Mx$

From (1) and (2) one has

if $M > 0$, $V(x) = x^T Mx$ is a positive definite (pd) function.

if $M \geq 0$, $V(x) = x^T Mx$ is a positive semi-definite (pd) function.

Lyapunov functions for LTI systems

Lyapunov functions for LTI systems

For linear system $\dot{x} = Ax$.
Consider as Lyapunov candidate function $V(x) = x^T P x$, $P = P^T > 0$.

$V(x)$ is quadratic, gpd and radially unbounded.

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T (A^T P + PA) x$

If $A^T P + PA < 0$, i.e. there is $Q > 0$ symmetric such that

$$A^T P + PA = -Q \tag{7}$$

then $\dot{V}$ is globally negative definite and by the second Lyapunov method the origin is globally asymptotically stable.

$A^T P + PA = -Q$ is called Lyapunov equation

Lyapunov theorem for LTI systems

Remarks

For LTI systems it is enough to consider quadratic Lyapunov functions.

Algorithm:

- choose $Q > 0$ (e.g. Q=I)
- solve $A^T P + PA = -Q$ (linear systems in the entries of the symmetric matrix P)
- The LTI system is AS if and only if $P > 0$

Example:

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix}}_{A} x$$

Eigenvalues of $A : \{-1, -3\} \Longrightarrow$ (global) asymptotic stability.

Choose $Q = Q^T = I_{2\times 2}$. Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$, where $p_{12} = p_{21}$.

Lyapunov theorem for LTI systems

Solve the Lyapunov equation $A^T P + PA = -Q$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Solving for p_{11}, p_{12} and p_{22} gives

$$\begin{cases} 2p_{11} &= -1 \\ -4p_{12} + 4p_{11} &= 0 \\ 8p_{12} - 6p_{22} &= -1 \end{cases}$$

Solving the linear systems one gets

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

Since $P > 0 \Longrightarrow$ the systems is AS.