Essentially Nonlinear Phenomena

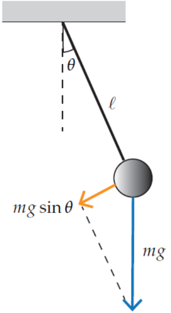
1. Multiple Isolated equilibrium

Linear systems have only one equilibrium point at origin or a number of non isolated equilibrium points.

Example: Pendulum (two isolated equilibia)

$$lm\ddot{\theta} = -kl\dot{\theta} - mg\sin(\theta) \qquad (6)$$

Define $x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$. State space : $S^1 \times \mathbb{R}$



$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_2 - \frac{g}{l}\sin x_1 \end{aligned} \qquad (7)$$

Equilibria: Two isolated E.P (0,0) and (π, 0).
Linearization :

$$\frac{\delta f}{\delta x} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos x_1 & -\frac{k}{l} \end{bmatrix} = \begin{cases} \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{l} \end{bmatrix} & \text{(stable) at } x_1 = 0 \\ \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{l} \end{bmatrix} & \text{(unstable) at } x_1 = \pi \end{cases} \qquad (8)$$

Stability may depend upon initial conditions

Example : The logistic equation (population dynamics in isolation)

$$\dot{x} = f(x) = \underbrace{rx(k-x)}_{\text{growth rate}} x, \qquad r > 0, K > 0 \tag{9}$$

$x \in \mathbb{R}_+$ is the population size and K is the carrying capacity.
Stability can be determined from the sign of $f(x)$ around the equilibrium.

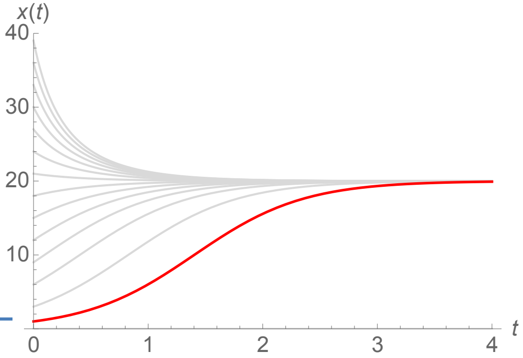
Equilibrium points are : $(\dot{x}_e = 0)$

$$\Rightarrow rx_e(K - x_e) = 0 \Rightarrow \begin{cases} x_e = 0 \\ x_e = K \end{cases}$$

Linearization :

$$\left.\frac{\partial f}{\partial x}\right|_{x_e} = (rK - 2rx)|_{x_e}$$

$$\begin{cases} \bar{x} = 0 \rightarrow Kr \rightarrow \text{unstable.} \\ \bar{x} = K \rightarrow -Kr \rightarrow \text{stable} \end{cases} \tag{10}$$



2. Finite Escape Time

In linear case :

$$\dot{x} = \lambda x \quad \xrightarrow{Solution} \quad x(t) = \exp^{\lambda t} x(0).$$

If $\lambda > 0 \Longrightarrow \lim_{t\to\infty} |x(t)| = +\infty$. Only as $t \to \infty$, $|x(t)| \to \infty$.

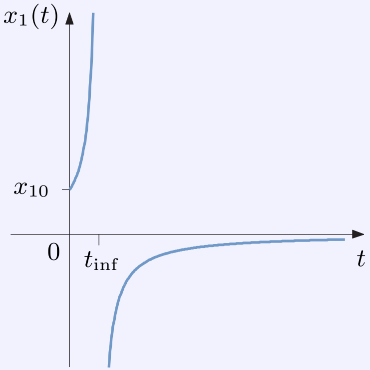
In nonlinear case :

Example : $\dot{x} = x^2$, $x(0) = x_0$, $x \in \mathbb{R}$

$$\frac{dx}{dt} = x^2 \quad \Rightarrow \quad \int_{x_0}^{x(t)} \frac{dx}{x^2} = \int_0^t dt$$
$$\Rightarrow \quad -\frac{1}{x(t)} + \frac{1}{x_0} = t - 0$$

$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

$x_0 > 0 \Rightarrow t \to \frac{1}{x_0} \Rightarrow x(t) \to \infty$



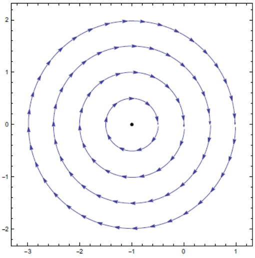
3. Limit cycles

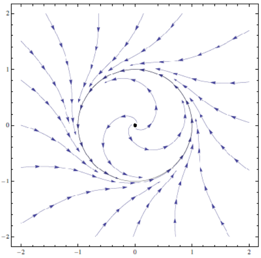
Linear oscillators exhibit a continuum of periodic orbits (closed orbit) :

$$x(t+T) = x(t),\ \forall t > 0, \text{for somme } T > 0$$

Every circle is periodic orbit for $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix}, \quad (\lambda_{1,2} = \pm j\beta)$$

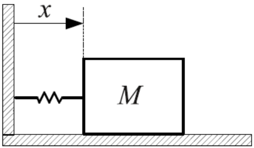




Every Periodic Orbit is a Cycle but not a Limit Cycle.
In contrast, a limit cycle is an isolated closed trajectory (closed orbit) and can occur only in nonlinear systems.

3. Limit cycles

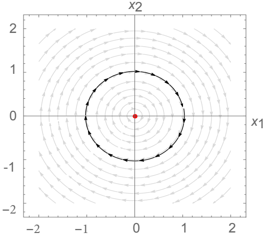
Example: Harmonic oscillator



$$\underbrace{m\ddot{x}}_{\text{inertial term}} + \underbrace{kx}_{\text{stiffness term}} = 0 \qquad (11)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\omega_0 = \sqrt{\frac{k}{m}}$. $(\lambda_{1,2} = \pm j\omega_0)$.



Amplitude of oscillations depends on initial conditions.

Can be destroyed by small modelling imperfections.

The harmonic oscillator has closed orbits but no limit cycles. Limit cycles cannot be generated by LTI systems.

The linear oscillator is not structurally stable. A stable oscillators must be produced by nonlinear systems.

3. Limit cycles

Example : Van der Pol oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

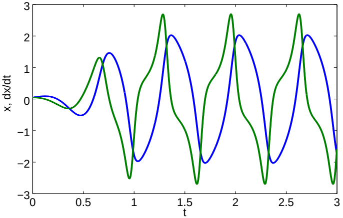
If $\mu = 0 \Rightarrow \ddot{x} + x = 0 \leftarrow$ simple harmonic oscillator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \mu(1 - x_1^2)x_2 - x_1 \end{cases} \quad (12)$$

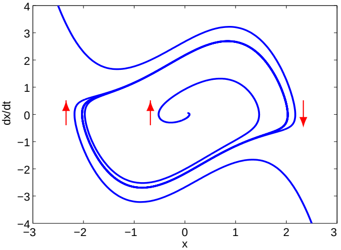
Equilibrium point :$\dot{x} = 0 \Rightarrow \bar{x} = [0\ 0]^T$.
Linarization around $\bar{x} = (0, 0)$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix}$$

Positive sign of μ tells us that e.p is unstable.

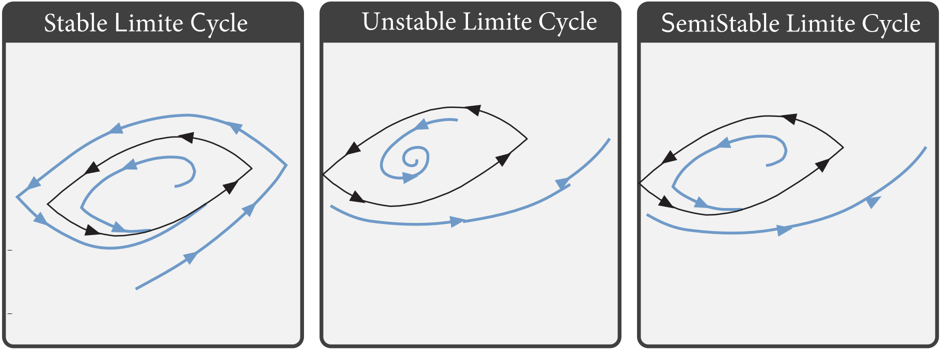


Response with $x(0) = 0.05$, $x^{\cdot}(0) = 0.05$



State trajectories $(x(t), \dot{x}(t))$

3. Limit cycles: Examples of limit cycles



4. Chaos :

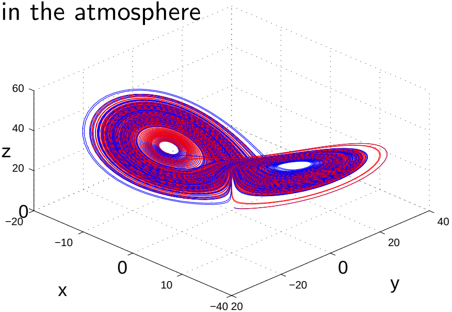
Chaos

Irregular oscillations, never exactly repeating.
Behavior of nonlinear systems may be extremely sensive to small changes in initial conditions/input/parameters.

Example : Lorenz system (attractor) derrived by Ed Lorenz in 1963 as a simplified model of convection rolls in the atmosphere

The lorenz system is a 3rd order system (3 states x,y,z).

$$\begin{aligned} \dot{x} &= \sigma(y-x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned} \qquad (13)$$



4. Chaos:

For continuous-time, time invariant systems, $n \geq 3$ state variable required for chaos.

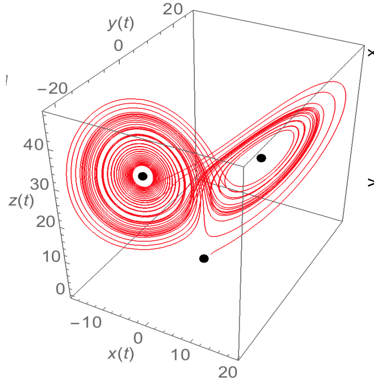
No simple characterization of asymptotic behavior.

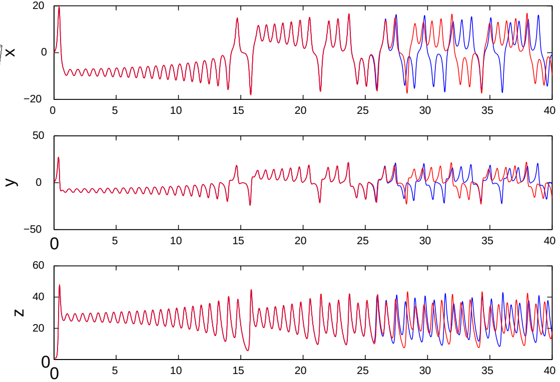
Huge sensitivity to initial conditions.

Choatic behavior with $\sigma = 10$, b=8/3, r=28

blue: $(x, y, z) = (0, 1, 1.05\ )$

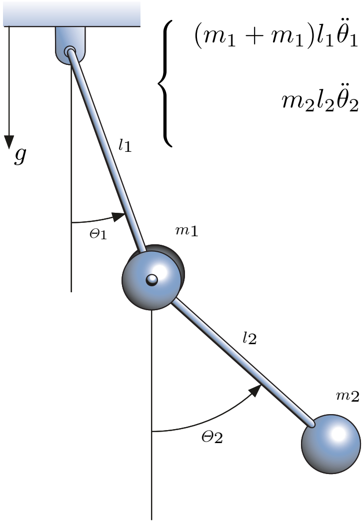
red: $(x, y, z) = (0, 1, 1.050001)$



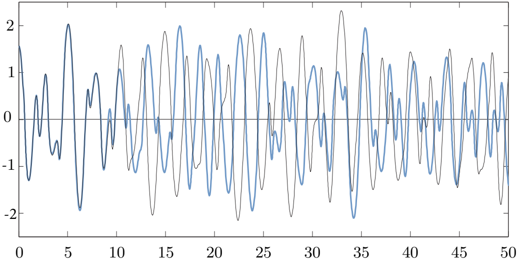


4. Chaos :

Example: The double pendulum (System is implicit for $l_1 \neq l_2$)



$$\begin{cases} (m_1+m_1)l_1\ddot{\theta}_1 & + \quad m_2l_2\ddot{\theta}_2\cos(\theta_1-\theta_2)+m_2l_2\dot{\theta}^2\sin(\theta_1-\theta_2) \\ & + \quad g(m_1+m_2)\sin\theta_1 = 0 \\ m_2l_2\ddot{\theta}_2 & + \quad m_2l_1\ddot{\theta}_1\cos(\theta_1-\theta_2)-m_2l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) \\ & + \quad m_2g\sin\theta_2 = 0 \end{cases}$$



Time t in s

5. Bifurcation: Fold bifurcation

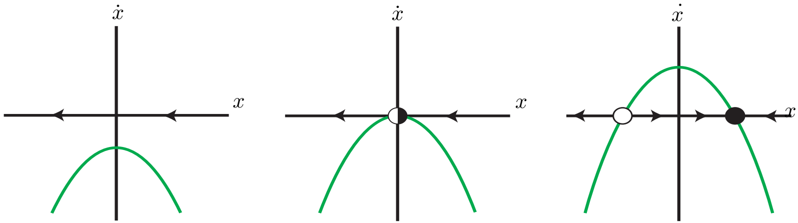
A **bifurcation** is an abrupt change in qualitative behavior as a parameter is varied. Examples : creation (or death) of equilibrium points (or limit cycles) and/or change of their stability properties.

Fold bifurcation: 1st order system

Example : $\dot{x} = \mu - x^2$,

Equilibrium points :

$$\bar{x} = \begin{cases} \pm\sqrt{\mu} & \mu > 0 \quad \text{one stable equilibrium and one unstable equilibrium} \\ 0 & \mu = 0 \quad \text{single equilibrium (called a saddle)} \\ \text{none} & \mu < 0 \quad \text{no equilibria} \end{cases}$$



5. Bifurcation: Fold bifurcation

$\mu_c = 0$ is the critical value of parameter μ which represents boundary between "no equilibrium points" and the presence of equilibrium points.

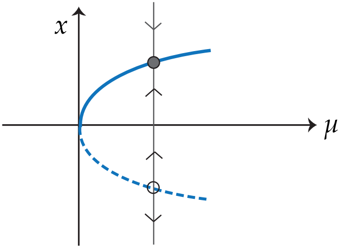
Creation/destruction of fixed points is called **saddle node bifurcation**

Linearization :

$$\left.\frac{\delta f}{\delta x}\right|_{\bar{x}} = 2\bar{x} = \begin{cases} 2\sqrt{\mu} & \text{unstable} \\ -2\sqrt{\mu} & \text{stable} \end{cases}$$

$$\mu > 0.$$

bifurcation diagram



Note:

$A_c = \left.\frac{\delta f}{\delta x}\right|_{\bar{x}_c = \bar{x}(\mu_c)} = 0 \rightarrow$ linearization disappears, no information about stability of the system.

5. Bifurcation: Transcritical bifurcation

- **Transcritical bifurcation:**

Example : $\dot{x} = \mu x - x^2, \quad x(t) \in \mathbb{R}$

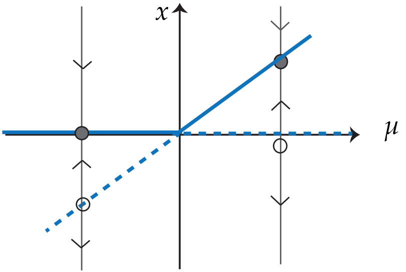
Equilibrium points : $\bar{x} = 0. \quad \bar{x} = \mu.$

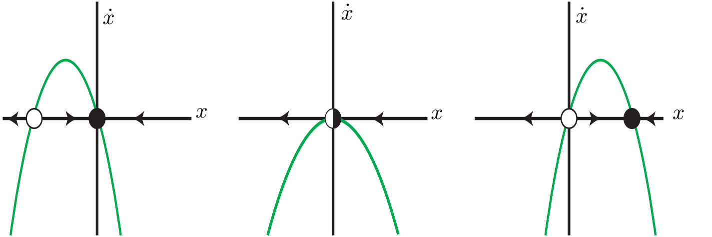
Linearization :

$$\frac{\delta f}{\delta x} = \mu - 2\bar{x} = \begin{cases} \mu & \text{if} \quad \bar{x} = 0 \\ -\mu & \text{if} \quad \bar{x} = \mu \end{cases}$$

$\mu < 0$: $\bar{x} = 0$ is stable, $\bar{x} = \mu$ is unstable.
$\mu > 0$: $\bar{x} = 0$ is unstable, $\bar{x} = \mu$ is stable.

bifurcation diagram





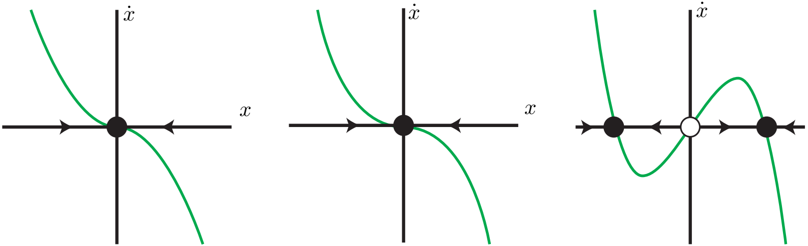
5. Bifurcation : Pichfork bifurcation

Pitchfork Bifurcation- 2 types : supercritical Pitchfork and subcritical pitchfork

Example 1:

$$\dot{x} \quad = \quad \mu x - x^3 \quad \text{supercritical}$$

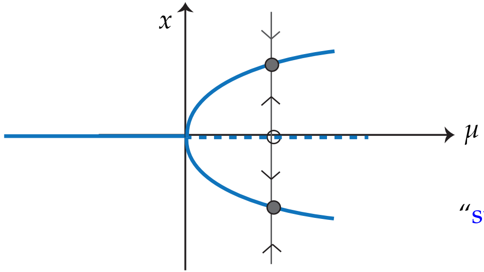
Equilibrium points : $f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{\mu}, \quad \mu > 0 \end{cases}$



2 equilibrium points emerge when we increase μ.

5. Bifurcation : Pitchfork bifurcation

bifurcation diagram



"supercritical pitchfork"

Example 2: $\dot{x} = \mu x + x^3$, subcritical pitchfork.
Equilibrium points :

$$f(\bar{x}) = 0 \Rightarrow \bar{x}(\mu - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{-\mu}, & \mu < 0 \end{cases}$$